Solving Problems in

GEOMETRY

by

V. Gusev

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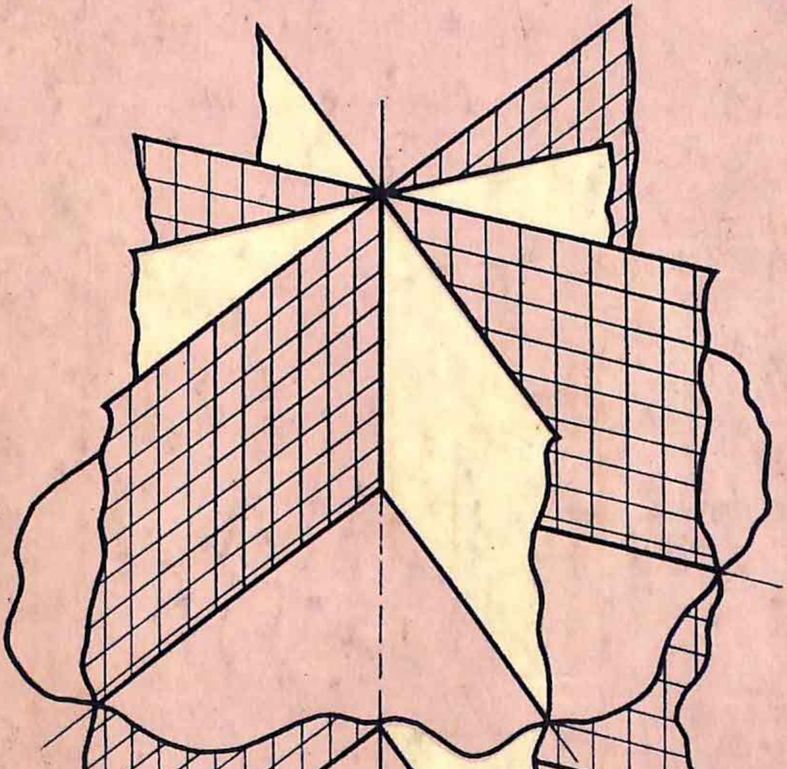
A. Mordkovich



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SOLVING PROBLEMS
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GEOMETRY

В. А. Гусев
В. Н. Литвиненко
А. Г. Мордкович

ПРАКТИКУМ ПО РЕШЕНИЮ МАТЕМАТИЧЕСКИХ ЗАДАЧ

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PREFACE

This book is intended for students at pedagogical (teacher training) institutes majoring in mathematics or in mathematics and physics. It has been written in correspondence with the current syllabus "Solving Problems".

When preparing the text, we wanted to represent the main types of problems in geometry found at school. The book contains about 1000 problems that should be solved independently. Alongside rather simple problems, there are problems whose solution requires profound meditation and sometimes even a nonstandard approach. The solution of most of the problems in this book will help the student form the professional habits important for a future teacher of mathematics, that is, to know how to solve the geometrical problems covered by the mathematics syllabus for high schools and vocational schools.

Various techniques and methods of solving geometrical problems are dealt with in the geometry courses at pedagogical institutes. However, not enough time is dedicated to the traditional methods of solution. To bridge this gap was one of the aims of our study aid.

We should like to emphasize that this book is not only a collection of problems, it is also a workbook in solving problems. This influences the contents and structure of the book itself.

Each section contains relevant theoretical material and detailed worked examples. We were especially careful when choosing the worked examples so that each solution will be helpful for the student, first from the methodological viewpoint, and so that the collection of these examples be ample and complete. Almost each of the sixteen sections has problems for the student to solve without assistance. They are grouped according to the sections and subsections of school geometry and in order of increasing difficulty. Most of the problems for independent solution are supplied with answers at the end of the book, a considerable number of problems have hints for their solution.

The present study aid consists of two chapters. Chapter 1 (Secs. 1-7) deals with planimetric problems, Sec. 1 being very important as it is, in a way, an introduction to the entire book. It discusses the methods for solving traditional geometrical problems, which will later be frequently used. Pure geometrical, algebraic, and combined methods are considered together with special cases, the method of a reference element (including the method of areas) and the method of an auxiliary parameter. To make the use of the book more convenient, this introductory section lists the important theorems of plane geometry, which should facilitate the solution of problems.

Sections 2 to 4 contain many standard problems of average difficulty, since experience shows that traditional planimetric problems are one of the weakest points in the preparation of future teachers of mathematics.

The main aim of Secs. 5 and 6 is to supply the student with the necessary habits and "know-how" for solving geometrical problems using the method of geometrical transformations and the vector method. We should like to underline here that these sections contain, as a rule, traditional geometrical problems to be solved by the indicated methods, but not the special problems on transformations and vectors that are frequently encountered in collections of problems in geometry. Since geometrical problems can be solved by different methods, the student will sometimes meet identical or similar problems in Secs. 2 to 4 and 5 to 6.

Two sections (7 and 16) are dedicated to the geometrical problems on finding the greatest and least values. These problems are usually thought to be a part of mathematical analysis, but in the latter the main purpose of these problems is to demonstrate the application of differential calculus (that is, the accent is on solving a problem within the framework of a mathematical model and, to a lesser extent, on setting up a model and interpreting it). When we included in this book problems for finding the greatest and least values, we were concerned that each problem should be interesting first from the geometrical viewpoint (that is, the accent was on the construction of a mathematical model and its interpretation).

Chapter 2 deals with stereometric problems. Questions on the construction of the representation of a given solid, and the determination of the completeness of the representation and its metric determinacy are posed in a concise form. Consideration is given to geometrical construction in space, particular attention being paid to constructions on representations. Most of the problems in both chapters were specially devised for this study aid. Among them, we should like to mention the determination of the angle between skew lines, the distance between them, the angle between a straight line and a plane, dihedral angles, and the construction of sections. In our opinion, solving these problems will help the student develop the ability of three-dimensional visualization.

The structure and contents of the book, the ways of setting forth the material, and the choice and arrangement of the problems were done by the authors collectively. The material for Secs. 1-4, 7 and

16 was prepared by A. G. Mordkovich, for Secs. 5 and 6 by V. A. Gusev, and for Secs. 8-15 by V. N. Litvinenko.

The authors are deeply greateful to the lecturers in algebra and geometry at the Ryazan State Pedagogical Institute, Assistant Professor M. M. Rassudovskaya, and G. A. Gal'perin, Cand. Sc. (Phys.-Math.), who read the manuscript attentively and made valuable suggestions that improved the book.

The Authors

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Chapter 1

PLANE GEOMETRY

SEC. 1. METHODS OF SOLVING GEOMETRICAL PROBLEMS

When solving geometrical problems, three basic methods are usually used: *geometrical* (a required statement is deduced from a number of known theorems with the aid of logical arguments), *algebraic* (a statement is proved or desired quantities are found by direct calculation based on various relations among geometrical quantities with the aid of setting up an equation or a system of equations), and *combined* (on some steps the solution is carried out by a geometrical method, on others by an algebraic method).

Whatever way of solution is chosen, its successful application depends on a knowledge of theorems and their use. Without citing here all the theorems of plane geometry (most of them are well known to the reader: such as tests for the congruence of arbitrary triangles, tests for the congruence of right triangles, basic properties of an isosceles triangle, parallelogram, rhombus, rectangle, the Thales theorem, the Pythagorean theorem, relationships between the sides and angles of a right triangle, tests for the similarity of triangles, theorem on the equality of arcs enclosed between parallel chords of a circle, etc.), we consider it necessary to recall the formulations of certain theorems often used when solving problems. Hereafter, references to these theorems will be made repeatedly.

I. Triangles and Quadrilaterals

1. Theorem on the equality of angles with mutually perpendicular sides: if $\angle ABC$ and $\angle DEF$ are both acute or both obtuse and $AB \perp DE$, $BC \perp EF$ (Fig. 1), then $\angle ABC = \angle DEF$.

2. Properties of the median (or midline) or a trapezoid:

(a) the median is parallel to the bases of a trapezoid;

(b) the median is equal to half the sum of the bases of a trapezoid;

(c) the median (and only this line) bisects any line segment enclosed between the bases of a trapezoid (Fig. 2).

These theorems hold true for the midline of a triangle as well if the triangle is regarded as a confluent (or degenerate) trapezoid, one of whose bases has a length equal to zero.

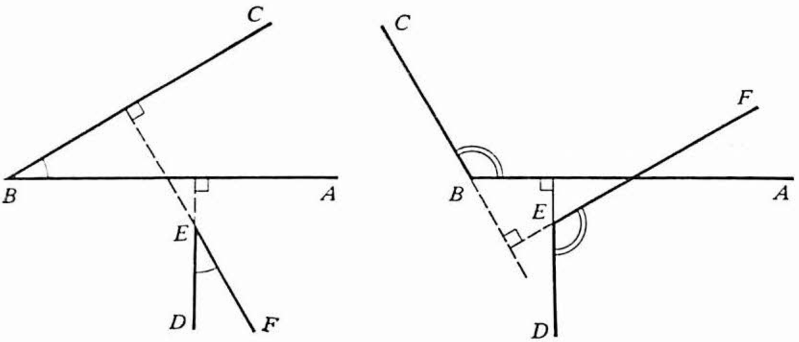

Fig. 1

3. Theorems on the points of intersection of the medians, bisectors, and altitudes (or heights) of a triangle:

(a) the three medians of a triangle are concurrent, that is, they intersect at a point which is the centroid (or the centre of gravity) of the triangle (sometimes called the median point), this point being two-thirds of the distance from a vertex to the opposite side along a median;

(b) the three bisectors of the angles of a triangle pass through a common point, which is equidistant from the sides of the triangle;

(c) the three altitudes of a triangle pass through a common point, which is called the orthocentre of the triangle.

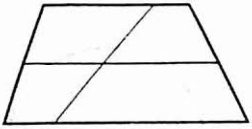

Fig. 2

4. Property of the median in a right triangle: in a right triangle, the median drawn to the hypotenuse is equal to half the hypotenuse. The converse is also true: if in a triangle, one of the medians is equal to half the side it is drawn to, then this is a right triangle.

5. Property of the bisector of an interior angle of a triangle: the bisector of an interior angle of a triangle divides the side to which it is drawn into parts proportional to the adjacent sides: $\frac{a}{b}=\frac{a'}{b'}$ (Fig. 3).

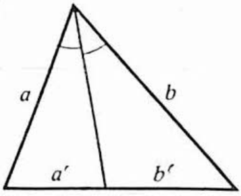

Fig. 3

6. Metric relationships in a right triangle: if a and b are legs, c hypotenuse, h height, and a' and b' projections of the legs on the hypotenuse (Fig. 4), then: (a) $h^2=a'b'$; (b) $a^2=ca'$; (c) $b^2=cb'$; (d) $a^2+b^2=c^2$; (e) $h=\frac{ab}{c}$.

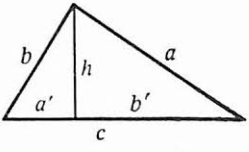

Fig. 4

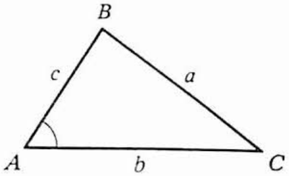

Fig. 5

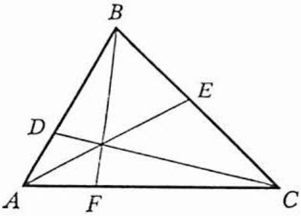

Fig. 6

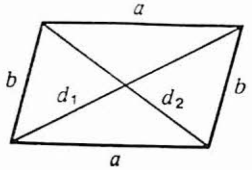

Fig. 7

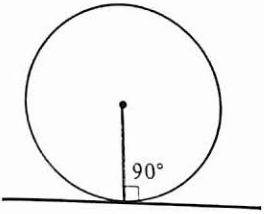

Fig. 8

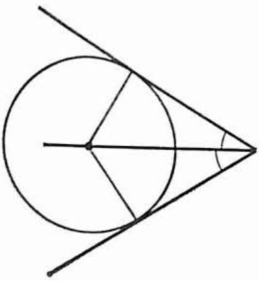

Fig. 9

7. Law of cosines: $a^2 = b^2 + c^2 - 2bc \cos A$ (Fig. 5).

8. Law of sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the radius of the circle circumscribed about the triangle.

9. Determining the kind of a triangle by its sides: let a, b, and c denote the sides of a triangle, c being the largest side, then:

(a) if $c^2 < a^2 + b^2$, then we have an acute triangle;

(b) if $c^2 = a^2 + b^2$, then we have a right triangle;

(c) if $c^2 > a^2 + b^2$, then we have an obtuse triangle.

10. *Ceva's theorem*: if three concurrent straight lines pass through the vertices A, B, and C of a triangle and intersect the opposite sides, produced if necessary at D, E, and F, then the product of the lengths of three alternate segments is equal to the product of the other three.

Let in the triangle ABC the points D, E, and F be taken on the sides AB, BC, and AC, respectively. For the lines AE, BF, and CD to be concurrent (Fig. 6), it is necessary and sufficient that the following equality be fulfilled:

$$\frac{AD}{BD} \cdot \frac{BE}{CE} \cdot \frac{CF}{AF} = 1.$$

11. Metric relationships in a parallelogram: the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of all of its sides: $d_1^2 + d_2^2 = 2a^2 + 2b^2$ (Fig. 7).

II. Circles

12. Properties of the lines tangent to a circle:

(a) the radius drawn to the point of tangency is perpendicular to the tangent (Fig. 8);

(b) two tangents drawn to a circle from an external point are equal, and make equal angles with the line joining that point to the centre (Fig. 9).

13. Angle measurement:

(a) a central angle equals in degrees its intercepted arc;

(b) an inscribed angle equals in degrees half its intercepted arc;

(c) an angle formed by a tangent and a chord equals in degrees half the intercepted arc.

14. Theorems on circles and triangles:

(a) a circle can be circumscribed about any triangle; the centre of the circumscribed circle lies at the point of intersection of the perpendiculars drawn to the sides through their midpoints;

(b) a circle can be inscribed in any triangle; the centre of the inscribed circle is the point of intersection of the angle bisectors of the triangle.

15. Theorems on circles and quadrilaterals:

(a) in order for a circle to be circumscribed about a quadrilateral, it is necessary and sufficient that the sum of its opposite angles be equal to 180° ($\alpha + \beta = 180°$, Fig. 10);

(b) in order for a circle to be inscribed in a quadrilateral, it is necessary and sufficient that the sums of its opposite sides be equal ($a + c = b + d$, Fig. 11).

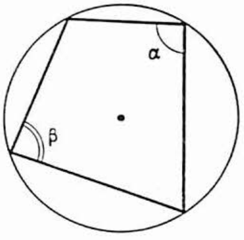

Fig. 10

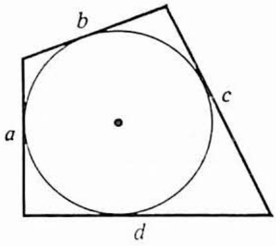

Fig. 11

16. Metric relationships in a circle:

(a) if two chords AB and CD intersect at the point M, then $AM \cdot BM = CM \cdot DM$ (Fig. 12);

(b) if two secants MAB and MCD are drawn to a circle from an external point M, then $AM \cdot BM = CM \cdot DM$ (Fig. 13);

(c) if a secant MAB and a tangent MC are drawn to a circle from an external point M, then $AM \cdot BM = CM^2$ (Fig. 14).

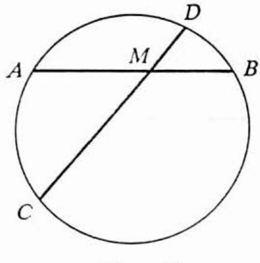

Fig. 12

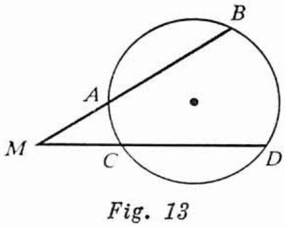

Fig. 13

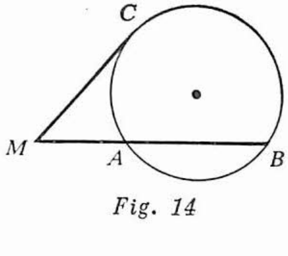

Fig. 14

III. Areas of Plane Figures

17. The ratio of the areas of similar figures is equal to the square of the ratio of similitude.

18. If two triangles have equal bases, then their areas are to each other as their altitudes; if two triangles have equal altitudes, then their areas are to each other as their bases.

19. Formulas for computing the area of a triangle:

(a) $S=\frac{ah}{2}$; (b) $S=\frac{ab\sin C}{2}$; (c) $S=\frac{abc}{4R}$; (d) $S=pr$, where $p=\frac{a+b+c}{2}$; R is the radius of the circumscribed circle, and r the radius of the inscribed circle.

(e) $S=\sqrt{p(p-a)(p-b)(p-c)}$ (*Hero's formula*).

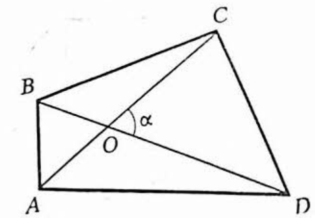

Fig. 15

20. Formulas for computing the area of a convex quadrilateral (Fig. 15):

(a) $S=S_{ABC}+S_{ACD}=S_{ABD}+S_{BCD}=S_{AOB}+S_{BOC}+S_{COD}+S_{AOD}$;

(b) $S=\frac{1}{2}AC\cdot BD\cdot\sin\alpha$;

(c) $S=pr$ (if a circle can be inscribed in a quadrilateral, and r is its radius).

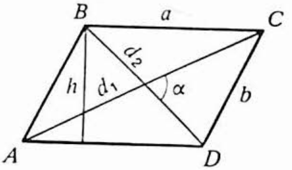

Fig. 16

21. Formulas for computing the area of a parallelogram (Fig. 16):

(a) $S=ah$; (b) $S=ab\sin C$; (c) $S=\frac{1}{2}d_1d_2\sin\alpha$.

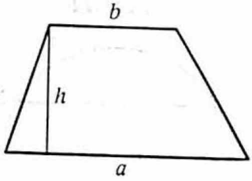

Fig. 17

22. Formula for the area of a trapezoid (Fig. 17): $S=\frac{a+b}{2}h$.

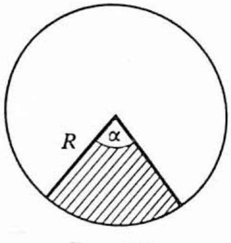

Fig. 18

23. Formula for the area of a sector of a circle (Fig. 18): $S=\frac{1}{2}R^2\alpha$ (α is a radian measure of the central angle).

24. Formula for the area of a segment of a circle (Fig. 19): $S=\frac{1}{2}R^2(\alpha-\sin\alpha)$.

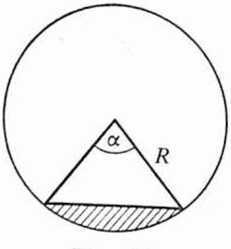

Fig. 19

When solving geometrical problems, we often have to ascertain the congruence of two line segments (or angles). Listed below are *three principal ways in which we prove geometrically that two line segments are congruent (equal in length)*:

(1) we regard the line segments as the sides of two triangles and prove that these triangles are congruent;

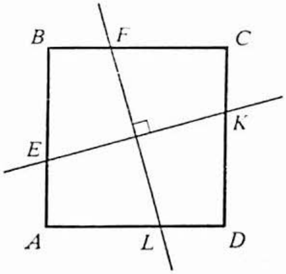

Fig. 20

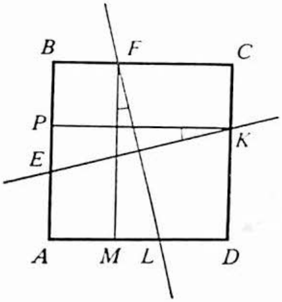

Fig. 21

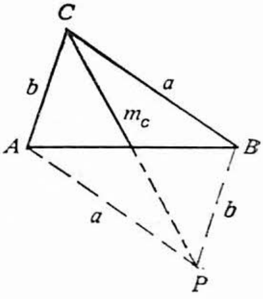

Fig. 22

(2) we regard the line segments as the sides of one triangle and prove that this triangle is isosceles;

(3) we replace the line segment a by a congruent line segment a', and the line segment b by a congruent line segment b' and prove that the line segments a' and b' are congruent.

When solving geometrical problems, we have to carry out auxiliary constructions, such as: drawing a straight line which is parallel or perpendicular to one shown in the figure; doubling the length of a median of a triangle thus completing the triangle to a parallelogram; describing an auxiliary circle; drawing radii to the points of tangency of a circle and a straight line or of two circles.

Example 1. Two mutually perpendicular lines intersect the sides AB, BC, CD, and AD of the square $ABCD$ at the points E, F, K, and L, respectively. Prove that $EK = FL$ (Fig. 20).

Solution. Using the first of the above ways, we draw FM parallel to CD and KP parallel to AD. Then the line segments EK and FL, we are interested in, will become sides of two right triangles EKP and FLM (Fig. 21), and, hence, it is sufficient to prove that these triangles are congruent.

We have: $PK = FM$ (as the altitudes of the given square), $\angle LFM = \angle EKP$ (as angles with mutually perpendicular sides, Theorem 1). Hence, the triangles EKP and FLM are congruent (as they have a respectively equal leg and an acute angle). The congruence of right triangles implies the congruence of their hypotenuses, that is, the line segments EK and FL.

Example 2. The sides of a triangle are equal to a, b, and c. Compute the median m_c drawn to the side c.

Solution. Extend the median to double it and construct the parallelogram $ACBP$ (Fig. 22). Applying Theorem 11 to this parallelogram, we get: $CP^2 + AB^2 = 2AC^2 + 2BC^2$, that is, $(2m_c)^2 + c^2 = 2b^2 + 2a^2$, whence we find that $m_c = \frac{\sqrt{2a^2+2b^2-c^2}}{2}$.

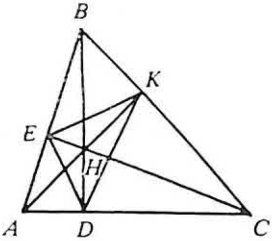

Fig. 23

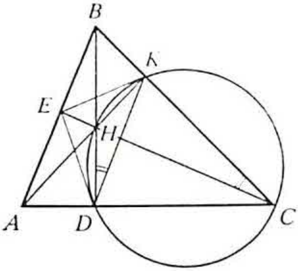

Fig. 24

Example 3. Prove that the orthocentre of an acute triangle coincides with the centre of the circle inscribed in the triangle formed by the feet of the altitudes.

Solution. Since the centre of a circle inscribed in a triangle lies at the point of intersection of the bisectors of the angles of the triangle (Theorem 14b), the problem is reduced to proving that DH, EH, and KH are bisectors of the angles of the triangle DEK (Fig. 23). For this purpose, it is sufficient to prove that $\angle EDH = \angle HDK$.

Consider the quadrilateral $DHKC$. We have: $\angle HDC = 90^\circ$ and $\angle HKC = 90^\circ$. Hence, $\angle HDC + \angle HKC = 180^\circ$, and, therefore, a circle can be circumscribed about the quadrilateral $DHKC$ (Theorem 15a).

Describing this circle (Fig. 24), we note that the angles HDK and HCK are congruent as inscribed angles subtended by one and the same arc HK. Analogously, circumscribing a circle about the quadrilateral $AEHD$, we conclude that $\angle EAH = \angle EDH$.

Thus, $\angle EAH = \angle EDH$ and $\angle HDK = \angle HCK$. But the angles EAH and HCK are congruent as angles with mutually perpendicular sides (Theorem 1), and, hence, $\angle EDH = \angle HDK$, which was required to be proved.

To set up equations in geometrical problems, we make use of the Pythagorean theorem, metric relationships in a right triangle (Theorem 6), relations between the sides and angles of a right triangle, proportionality of the sides, altitudes, and perimeters of similar triangles, the property of the bisector of the angles of a triangle (Theorem 5), metric relationships in a parallelogram (Theorem 11) and a circle (Theorem 16), law of sines (Theorem 8), law of cosines (Theorem 7), and various formulas for computing areas.

The *method of a reference element* is the fundamental method of setting up an equation in solving a geometrical problem. This method consists in the following: one and the same element is expressed (in terms of known and unknown quantities) in two different ways, and the expressions thus obtained are equated to each other. If area is used as a reference element, then the problem is said to be solved by the *method of areas*.

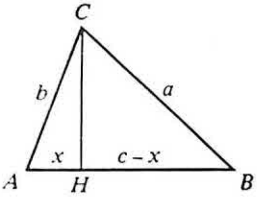

Fig. 25

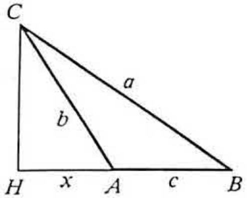

Fig. 26

Example 4. The sides of a triangle are equal to a, b, and c. Compute the altitude h_c drawn to the side c.

Solution. First Method. The altitude h_c is a common leg of two right triangles ACH and CHB (Fig. 25). Taking advantage of the Pythagorean theorem, we express CH^2 from the triangles ACH and CHB (CH being a reference element).

Let $AH = x$, then $BH = c - x$. If the triangle ACB were obtuse, then it would be $BH = c + x$ (Fig. 26). Here we confine ourselves to the case represented in Fig. 25.

We find from the triangle ACH that $CH^2 = b^2 - x^2$, and from the triangle BCH that $CH^2 = a^2 - (c - x)^2$. We find from the equation $b^2 - x^2 = a^2 - (c - x)^2$ that $x = \frac{c^2 + b^2 - a^2}{2c}$

We obtain the following formula from the triangle ACH:

$$CH = \sqrt{b^2 - \left(\frac{c^2 + b^2 - a^2}{2c}\right)^2}$$

$$= \sqrt{\left(b - \frac{c^2 + b^2 - a^2}{2c}\right)\left(b + \frac{c^2 + b^2 - a^2}{2c}\right)}$$

$$= \frac{1}{2c}\sqrt{(a^2 - (b - c)^2)\,((b + c)^2 - a^2)}$$

$$= \frac{1}{2c}\sqrt{(a + b - c)\,(a + c - b)\,(b + c - a)\,(a + b + c)}\,.$$

Thus, $h_c = \frac{\sqrt{(a+b+c)\,(a+b-c)\,(a+c-b)\,(b+c-a)}}{2c}$.

Second Method. Let us use the method of areas. On the one hand, the area of the triangle ABC is equal to $\sqrt{p\,(p-a)(p-b)(p-c)}$, and, on the other hand, it is equal to $\frac{1}{2}ch_c$. Equating these expressions, we get:

$$h_c = \frac{2\sqrt{p\,(p-a)\,(p-b)\,(p-c)}}{c}$$

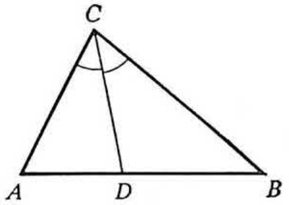

Fig. 27

Substituting the expression for p in terms of sides, that is, $p=\frac{a+b+c}{2}$, we get:

$$h_c=\frac{\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}}{2c}.$$

Example 5. Given the sides of a triangle (a, b, and c). Find the angle bisector l_c drawn to the side c.

Solution. First Method (algebraic). Let CD be an angle bisector of the triangle ABC (Fig. 27). The scheme of solution is the following: we find the lengths of the line segments AD and BD, and then, applying the law of cosines to the triangles ACD and BCD (bearing in mind that $\angle ACD = \angle DCB$), we find the angle bisector $l_c = CD$.

Let $AD = x$ and $BD = y$. Then $x + y = c$, and by the property of an angle bisector (Theorem 5), we have: $\frac{x}{y}=\frac{b}{a}$. We find from the system of equations $\begin{cases} x+y=c, \\ \frac{x}{y}=\frac{b}{a} \end{cases}$ that $x=\frac{bc}{a+b}$, $y=\frac{ac}{a+b}$.

Applying the law of cosines (Theorem 7) to the triangle ACD, we get:

$$x^2 = b^2 + l^2 - 2bl \cos t \tag{1}$$

(for the sake of brevity, we have put here that $l_c = l$ and $\angle ACD = \angle DCB = t$).

Applying the law of cosines to the triangle BCD, we get:

$$y^2 = a^2 + l^2 - 2al \cos t. \tag{2}$$

We multiply both sides of Equality (1) by a, and both sides of Equality (2) by $(-b)$, and add together the equalities thus obtained: $ax^2 - by^2 = ab^2 - a^2b + al^2 - bl^2$, whence we find:

$$l^2=\frac{1}{a-b}(x^2a-y^2b)+ab. \tag{3}$$

Substituting the values of x and y found above into Equality (3), we get: $l^2=\frac{1}{a-b}\left(\frac{b^2c^2a}{(a+b)^2}-\frac{a^2c^2b}{(a+b)^2}\right)+ab=ab\left(1-\frac{c^2}{(a+b)^2}\right)=\frac{ab(a+b+c)(a+b-c)}{(a+b)^2}$.

Thus, $l_c=\frac{\sqrt{ab(a+b+c)(a+b-c)}}{a+b}$.

Second Method. In addition to the sought-for unknown l, let us introduce an auxiliary unknown quantity: we set $x = \angle ACD =$

$\angle DCB$ and use the method of areas. We have: $S_{ABC} = S_{ACD} + S_{BCD}$. On the one hand, $S_{ABC} = \frac{1}{2} ab \sin 2x$. On the other hand, since $S_{ACD} = \frac{1}{2} bl \sin x$ and $S_{BCD} = \frac{1}{2} al \sin x$, we have: $_{ABC} = \frac{1}{2} al \sin x + \frac{1}{2} bl \sin x$.

Hence, $\frac{1}{2} ab \sin 2x = \frac{l(a+b)\sin x}{2}$, whence $l = \frac{2ab \cos x}{a+b}$.

To find $\cos x$, let us apply the law of cosines to the triangle ABC for the side AB. We get: $c^2 = a^2 + b^2 - 2ab \cos 2x$, whence we find that $\cos 2x = \frac{a^2+b^2-c^2}{2ab}$. Then $\cos x = \sqrt{\frac{1+\cos 2x}{2}} =$

$$\sqrt{\frac{1}{2}\left(1+\frac{a^2+b^2-c^2}{2ab}\right)} = \frac{1}{2}\sqrt{\frac{(a+b+c)(a+b-c)}{ab}}.$$

Thus, we get:

$$l = \frac{2ab \cos x}{a+b} = \frac{2ab}{a+b} \cdot \frac{1}{2}\sqrt{\frac{(a+b+c)(a+b-c)}{ab}}$$
$$= \frac{\sqrt{ab(a+b+c)(a+b-c)}}{a+b}.$$

When setting up equations in the process of solution of a geometrical problem, the progress in solution often depends on a successful introduction of unknowns. Let us clarify this by the following example.

Example 6. In a right triangle, the hypotenuse is equal to c, and the bisector of one of the acute angles equals $\frac{c\sqrt{3}}{3}$. Find the legs (Fig. 28).

Solution. First Method. We set $AC = x$, $BC = y$, and $CD = z$. Then, by the Pythagorean theorem, $x^2 + y^2 = c^2$ and $x^2 + z^2 = \left(\frac{c\sqrt{3}}{3}\right)^2$. Besides, by the property of an angle bisector (Theorem 5), we have: $\frac{AC}{AB} = \frac{CD}{DB}$, i.e. $\frac{x}{c} = \frac{z}{y-z}$.

In the long run, we get the following system of three equations in three variables:

$$\begin{cases} x^2 + y^2 = c^2, \\ x^2 + z^2 = \dfrac{c^2}{3}, \\ \dfrac{x}{c} = \dfrac{z}{y-z}, \end{cases}$$

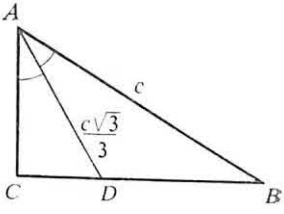

Fig. 28

whose solution involves considerable algebraic difficulties.

Second Method. We put $\angle CAD = \angle BAD = x$. Let us set up an equation, using the line segment AC as a reference element. We find from the triangle ABC that $AC = c\cos 2x$, and from the triangle ACD that $AC = \frac{c\sqrt{3}}{3}\cos x$. Equating these expressions, we get the following trigonometric equation: $c\cos 2x = \frac{c\sqrt{3}}{3}\cos x$.

Let us solve this equation: $\sqrt{3}\cos 2x = \cos x$, $\sqrt{3}(2\cos^2 x - 1) = \cos x$, $2\sqrt{3}\cos^2 x - \cos x - \sqrt{3} = 0$, whence $\cos x = \frac{\sqrt{3}}{2}$, or $\cos x = -\frac{\sqrt{3}}{3}$. Since, according to the sense of the problem, $\cos x > 0$, we get: $\cos x = \frac{\sqrt{3}}{2}$. Hence, the angle BAD is equal to 30°, and the angle BAC to 60°. As a result, we find that $AC = \frac{c}{2}$ and $BC = \frac{c\sqrt{3}}{2}$.

If in a problem it is required to find the ratio of certain quantities (of lengths or areas), in particular, if it is required to compute an angle (which is usually reduced to finding a certain trigonometric function of the angle, and, hence, to finding the ratio of the sides of a right triangle), we usually proceed as follows: we assume one of the linear elements to be known, express the desired quantities in terms of this element, and then form their ratio. The introduced linear element is called an auxiliary parameter, and this method of solving geometrical problems is called the *method of introducing an auxiliary parameter*. It is used in solving problems where the geometrical figures are defined to within similarity.

Example 7. The altitude and a median are drawn from the right angle C of a right triangle ABC. The angle α between them is equal to $\arccos\frac{40}{41}$. Find the ratio of the legs (Fig. 29).

Solution. Let us first of all note that, by the hypothesis, $\cos\alpha = \frac{40}{41}$, that is, $\frac{CK}{CM} = \frac{40}{41}$. We are going to solve the problem by the method of introducing an auxiliary parameter.

Let us set $CK = h$. Then $CM = \frac{41}{40}h$ and $KM = \sqrt{CM^2 - CK^2} = \frac{9}{40}h$. Since the median is equal to half the hypotenuse in a right triangle (Theorem 4), we have: $AM = CM = MB = \frac{41}{40}h$. Then $AK = AM - KM = \frac{41}{40}h - \frac{9}{40}h = \frac{4}{5}h$, $KB = KM + BM = \frac{9}{40}h + \frac{41}{40}h =$

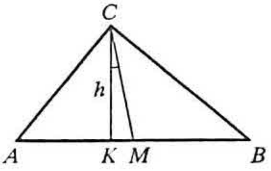

Fig. 29

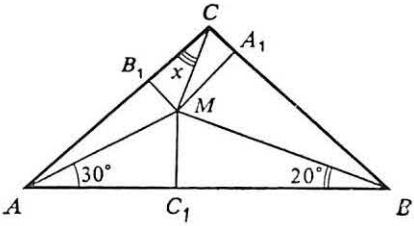

Fig. 30

$\frac{5}{4}h$, $AC=\sqrt{AK^2+CK^2}=\sqrt{\frac{16}{25}h^2+h^2}=\frac{h}{5}\sqrt{41}$, $BC=\sqrt{BK^2+CK^2}=\sqrt{\frac{25}{16}h^2+h^2}=\frac{h}{4}\sqrt{41}$, and $\frac{AC}{BC}=\frac{h\sqrt{41}}{5}\div\frac{h\sqrt{41}}{4}=\frac{4}{5}$.

Example 8. The angle C at the vertex of an isosceles triangle ABC is equal to 100°. Two rays are constructed: one—with the origin at the point A and at an angle of 30° to the ray AB, and the other—with the origin at the point B and at an angle of 20° to the ray BA. The rays intersect at the point M. Find the angles ACM and BCM (Fig. 30).

Solution. We connect the points M and C, and denote the angle ACM by x. We then drop perpendiculars from the point M to the sides of the triangle: $MC_1 \perp AB$, $MB_1 \perp AC$, and $MA_1 \perp BC$. We introduce an auxiliary parameter by setting $CM = a$ and compute MC_1 in two ways, that is, by using MC_1 as a reference element.

We find from the triangle CMB_1 that $MB_1 = MC \sin x = a \sin x$. Since $\angle ACB = 100°$, and the triangle ABC is, by the hypothesis, isosceles, we have: $\angle CAB = \angle ABC = 40°$, and, hence, $\angle CAM = 10°$. We find from the triangle AMB_1 that $AM = \frac{MB_1}{\sin 10°} = \frac{a \sin x}{\sin 10°}$. Finally, we obtain from the triangle AMC_1: $MC_1 = AM \sin 30° = \frac{a \sin x}{2 \sin 10°}$.

Consider the triangle CMA_1. In it, $\angle MCA_1 = 100° - x$. Hence, $MA_1 = CM \sin(100° - x) = a \sin(100° - x)$. Since $\angle MBC = 40° - 20° = 20°$, the triangles BMC_1 and BMA_1 are congruent, and, therefore, $MC_1 = MA_1 = a \sin(100° - x)$. Equating the expressions found for MC_1, we get the trigonometric equation: $\frac{a \sin x}{2 \sin 10°} = a \sin(100° - x)$. From this equation we find in succession: $\sin x = 2 \sin(100° - x) \cdot \sin 10°$, $\sin x = \cos(90° - x) - \cos(110° - x)$, $\cos(110° - x) = 0$, and, hence, $x = 20°$.

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SEC. 2. TRIANGLES AND QUADRILATERALS

Example 1. Given the bases a and b of a trapezoid. Find the length of the line segment joining the midpoints of the diagonals (Fig. 31).

Solution. Since the point P is the midpoint of the diagonal AC, and the point K is the midpoint of the diagonal BD, the points P and K lie on the median EF of the trapezoid (Theorem 2c). Since EK is the midline of the triangle ABD, we have: $EK = \frac{a}{2}$, and since EP is the midline of the triangle ABC, we have: $EP = \frac{b}{2}$. Thus, we obtain: $PK = EK - EP = \frac{a-b}{2}$.

Example 2. Knowing the medians m_a, m_b, and m_c of a triangle ABC, find the side $AC = b$.

Solution. According to Theorem 3a, the medians of a triangle intersect at a common point M. This point is two-thirds of the distance from a vertex to the opposite side along a median (Fig. 32). Therefore, two sides are known in the triangle AMC, i.e. $AM = \frac{2}{3} m_a$, $MC = \frac{2}{3} m_c$, and the median $MD = \frac{1}{3} m_b$.

Consider the triangle AMC. Doubling in length its median MD (the line segment MP) and joining the point P to the points A and C, we obtain a parallelogram $AMCP$ (Fig. 33). Then, by Theorem 11, $AC^2 + MP^2 = 2AM^2 + 2MC^2$, that is, $b^2 + \frac{4}{9} m_b^2 = 2\cdot\frac{4}{9} m_a^2 + \ \cdot\frac{4}{9} m_c^2$, whence we find that $b = \frac{2}{3}\sqrt{2m_a^2 + 2m_c^2 - m_b^2}$.

Example 3. Constructed on the sides AB and BC of a triangle ABC are squares $ABDE$ and $BCKM$. Prove that the line segment DM is twice the length of the median BP of the triangle ABC (Fig. 34).

Solution. Since we have to prove that $DM = 2BP$, it is expedient to double the length of the median BP, to transform the triangle ABC into a parallelogram $ABCT$, and then to prove that $DM = BT$

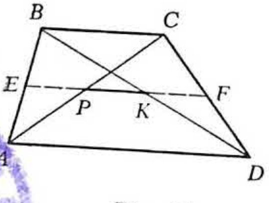


Fig. 31

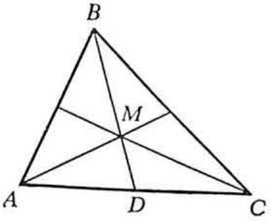


Fig. 32

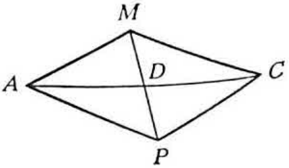


Fig. 33

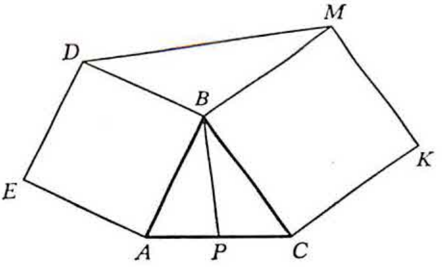


Fig. 34

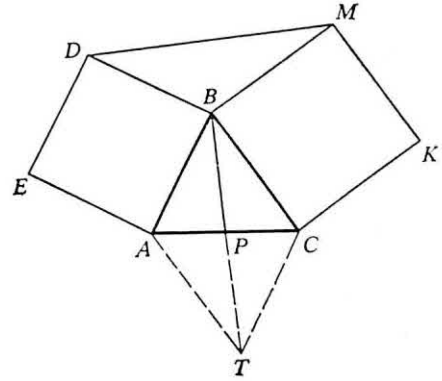


Fig. 35

(Fig. 35). To prove the congruence of the line segments DM and BT, we have to consider these segments as sides of two triangles and to prove that these triangles are congruent. According to this scheme, let us perform the solution of the given problem.

Laying off the line segment BP (whose length is equal to the length of the median BP) and joining the point T to the points A and C, we get a parallelogram $ABCT$.

Consider the triangles DMB and BCT. We have: (a) $BM=BC$ as sides of the square $BMKC$; (b) $DB = CT$ (in more detail: $DB=AB$ as sides of the square $BDEA$, and $AB = CT$ as the opposite sides of the parallelogram $ABCT$, and, hence, $DB=CT$; (c) $\angle DBM=\angle BCT$ (as angles with mutually perpendicular sides). Hence, $\triangle DBM=\triangle BCT$ (by two sides and the included angle), and, therefore, $DM=BT$. Since $BT = 2BP$, from $DM = BT$ it follows that $DM = 2BP$.

Example 4. The altitude drawn to the hypotenuse of a right triangle divides it into two parts, 9 cm and 16 cm long. From the vertex of the larger acute angle of the triangle, a straight line is drawn passing through the midpoint of the altitude. Find the length of the segment of this line enclosed inside the given right triangle (Fig. 36).

Solution. We have: $CH^2=AH\cdot BH$ (Theorem 6a), and, hence, $CH^2=9\cdot 16$, i.e. $CH=12$ cm. We find from $\triangle ADH$ that $AD=\sqrt{9^2+6^2}=3\sqrt{13}$ cm. We draw $HM \parallel AK$ and set $DK=x$. Since DK is the midline of $\triangle HCM$, we have: $HM=2x$. Since the triangles HMB and AKB are similar, we conclude that $\frac{HM}{AK}=\frac{BH}{AB}$, i.e. $\frac{2x}{x+3\sqrt{13}}=\frac{16}{25}$, whence $x=\frac{24\sqrt{13}}{17}$ and $AK=3\sqrt{13}+\frac{24\sqrt{13}}{17}=\frac{75\sqrt{13}}{17}$. Thus, $AK=\frac{75\sqrt{13}}{17}$ cm.

Example 5. A rectangle is inscribed in a triangle with sides equal to 10 cm, 17 cm, and 21 cm so that two of its vertices are situated on one side of the triangle, and two others on two other sides of the

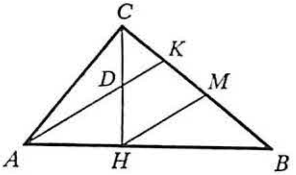


Fig. 36

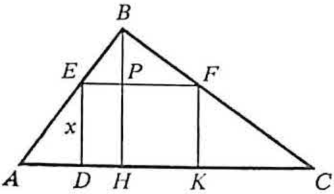


Fig. 37

triangle. Find the sides of the rectangle if it is known that its perimeter equals 22.5 cm.

Solution. Let us first of all determine the kind of the given triangle. We have: $10^2 = 100$, $17^2 = 289$, and $21^2 = 441$. Since $21^2 > 10^2 + 17^2$, the triangle is obtuse (Theorem 9), and, hence, a rectangle can be inscribed in it only in one way: by arranging two of its vertices on the larger side (Fig. 37).

We then find the altitude BH of the triangle ABC. Applying the method used in Example 4 from Sec. 1 (or using the formula derived in this example), we find that $BH = 8$ cm.

Setting $ED = x$, we get: $EF = 11.25 - x$ (since the perimeter of the rectangle $DEFK$ is equal to 22.5 cm) and $BP = 8 - x$.

The triangles BEF and ABC are similar. Hence, $\frac{EF}{AC} = \frac{BP}{BH}$ (in similar triangles the ratio of the corresponding altitudes is equal to the ratio of similitude), that is, $\frac{11.25-x}{21} = \frac{8-x}{8}$, whence we find that $x = 6$.

Thus, the sides of the rectangle are 6 cm and 5.25 cm.

Example 6. In a triangle ABC, the angle A is twice the angle C, the side BC is 2 cm longer than the side AB, and $AC = 5$ cm. Find AB and BC.

Solution. First Method. Drawing the bisector AD of the angle A, we get: $\angle BAD = \angle DAC = \angle ACB$ (Fig. 38).

In a triangle ADC, the base angles are equal to each other, and, hence, this is an isosceles triangle: $AD = DC$. Setting $AB = x$ and $AD = DC = y$, we find that $BC = x + 2$ and $BD = x + 2 - y$.

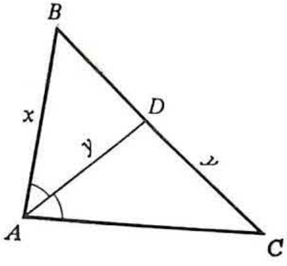


Fig. 38

The triangles ABD and ABC are similar, since $\angle BAD = \angle BCA$ and $\angle B$ is a common angle. From the similarity of these triangles we conclude that $\frac{AB}{BC} = \frac{BD}{AB} = \frac{AD}{AC}$, i.e. $\frac{x}{x+2} = \frac{x+2-y}{x} = \frac{y}{5}$.

For finding x and y, we have obtained a system of two equations in two variables: $\begin{cases} \dfrac{x}{x+2}=\dfrac{y}{5}, \\ \dfrac{x+2-y}{x}=\dfrac{y}{5}, \end{cases}$ whence $\begin{cases} 5x=xy+2y, \\ 5x+10-5y=xy. \end{cases}$ Subtracting the second equation from the first, we get: $5y-10=2y$ and $y=\frac{10}{3}$. Hence, $\frac{x}{x+2}=\frac{2}{3}$, i.e. $x=4$.

Thus, $AB=4$ cm and $BC=6$ cm.

Second Method. We set $\angle C=t$, then $\angle A=2t$ and $\angle B=180°-3t$. We also set $AB=x$, then $BC=x+2$. By the law of sines (Theorem 8), we have: $\frac{x}{\sin t}=\frac{x+2}{\sin 2t}=\frac{5}{\sin(180°-3t)}$.

Thus, we have obtained the following system of two equations in two variables x and t: $\begin{cases} \dfrac{x}{\sin t}=\dfrac{x+2}{\sin 2t}, \\ \dfrac{x}{\sin t}=\dfrac{5}{\sin 3t} \end{cases}$ (here we have taken advantage of the fact that $\sin(180°-3t)=\sin 3t$).

Let us solve this system. We get from the second equation: $x=\frac{5\sin t}{\sin 3t}=\frac{5\sin t}{3\sin t-4\sin^3 t}=\frac{5}{3-4\sin^2 t}$. We obtain from the first equation: $\frac{x+2}{x}=\frac{\sin 2t}{\sin t}$, i.e. $1+\frac{2}{x}=2\cos t$. Substituting instead of x its expression in terms of t, we get: $1+\frac{6-8\sin^2 t}{5}=2\cos t$. Setting in this trigonometric equation $\cos t=z$, we obtain: $1+\frac{6-8(1-z^2)}{5}=2z$, whence $z_1=\frac{3}{4}$ and $z_2=\frac{1}{2}$, that is, either $\cos t=\frac{3}{4}$ or $\cos t=\frac{1}{2}$.

If $\cos t=\frac{3}{4}$, then from $1+\frac{2}{x}=2\cos t$ we find that $x=4$.

If $\cos t=\frac{1}{2}$, then from $1+\frac{2}{x}=2\cos t$ we find that $1+\frac{2}{x}=1$, which is impossible.

Thus, $AB=4$ cm and $BC=6$ cm.

Remark. The expression $\cos t=\frac{1}{2}$ means that $t=60°$, then we obtain that in the triangle ABC, $\angle C=60°$ and $\angle A=120°$, which is impossible.

Example 7. The point K is the midpoint of the side AD of the rectangle $ABCD$ (Fig. 39). Find the angle between BK and the diagonal AC if it is known that $AD:AB=\sqrt{2}$.

Solution. Let us use the method of an auxiliary parameter (see Example 7 from Sec. 1). Setting $AB=a$, we find that $AD=a\sqrt{2}$. The solution scheme is the following: we first express all the sides of the triangle AMK in terms of a and then apply the law of cosines

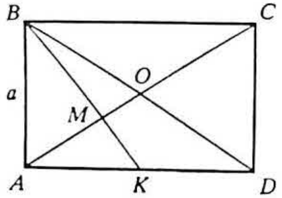


Fig. 39

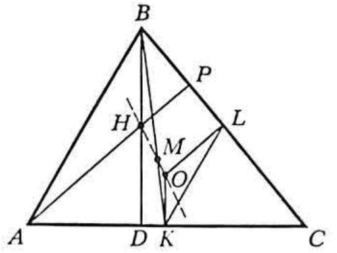


Fig. 40

to the side AK. This will make it possible to compute the cosine of the sought-for angle AMK denoted by x.

The line segments AO and BK are medians of the triangle ABD. Hence, $MK=\frac{1}{3}\,BK$ and $AM=\frac{2}{3}\,AO$ (Theorem 3a). We have: $MK=\frac{1}{3}\,BK=\frac{1}{3}\sqrt{AB^2+AK^2}=\frac{1}{3}\sqrt{a^2+\left(\frac{a\sqrt{2}}{2}\right)^2}=\frac{a\sqrt{6}}{6}$ and $AM=\frac{2}{3}AO=\frac{1}{3}AC=\frac{1}{3}\sqrt{AD^2+CD^2}=\frac{1}{3}\sqrt{(a\sqrt{2})^2+a^2}=\frac{a\sqrt{3}}{3}$.

In the triangle AMK we have: $AK=\frac{a\sqrt{2}}{2}$, $AM=\frac{a\sqrt{3}}{3}$, and $MK=\frac{a\sqrt{6}}{6}$. By the law of cosines (Theorem 7), $AK^2=AM^2+MK^2-2AM\cdot MK\cdot\cos x$, i.e. $\left(\frac{a\sqrt{2}}{2}\right)^2=\left(\frac{a\sqrt{3}}{3}\right)^2+\left(\frac{a\sqrt{6}}{6}\right)^2-2\cdot\frac{a\sqrt{3}}{3}\cdot\frac{a\sqrt{6}}{6}\cos x$, and, further, $\frac{a^2}{2}=\frac{a^2}{3}+\frac{a^2}{6}-\frac{a^2\sqrt{2}}{3}\cos x$, whence we find that $\cos x=0$, and, consequently, $x=90°$.

Thus, BK and AC are at right angles.

Example 8. Prove that in any triangle ABC, the distance from the orthocentre to the vertex B is twice the distance from the centre of the circle circumscribed about the triangle to the side AC.

Solution. Let ABC denote an acute triangle, H the orthocentre, O the centre of the circumscribed circle, BD and AP altitudes, K and L the midpoints of the sides, and OK and OL perpendiculars to the sides (Fig. 40).

The triangles ABH and KOL are similar ($BH \parallel OK$, $AH \parallel OL$, $AB \parallel LK$), and, hence, $\frac{BH}{OK}=\frac{AB}{LK}$. The line segment LK is the midline of $\triangle ABC$, and, hence, $\frac{AB}{LK}=2$. But then $\frac{AB}{OK}=2$, which was required to be proved.

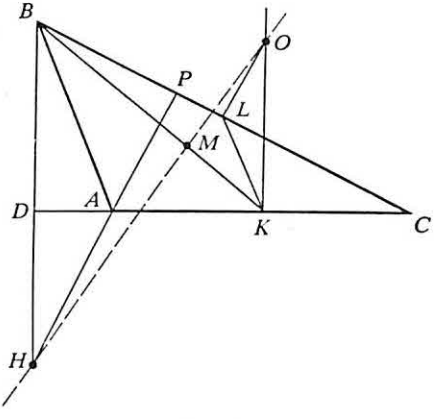


Fig. 41

Let ABC be an obtuse nonisosceles triangle, the notation being the same as in the preceding case (Fig. 41). It follows from the similarity of the triangles ABH and KOL that $\frac{BH}{OK} = \frac{AB}{LK} = 2$, and, hence, $BH = 2OK$.

Shown in Figs. 40 and 41 is Euler's line (drawn in broken line) (see Exercise 54).

Example 9. A straight line is drawn through the centroid of a regular triangle in the plane of this triangle. Prove that the sum of the squares of the distances from the vertices of the triangle to this line is independent of the choice of this line.

Solution. Let the straight line under consideration form an angle α with the base AC of the triangle ABC (Fig. 42). We set $AO = BO = CO = a$, express the perpendiculars AD, BK, and CE to this line in terms of a and α, and then prove that the expression $AD^2 + BK^2 + CE^2$ is constant for any α.

Since $\angle OAC = 30°$, we have: $\angle MAO = 150°$, then $\angle DOA = 180° - (\alpha + 150°) = 30° - \alpha$. We find from $\triangle DOA$ that $AD = OA \sin \angle AOD = a \sin (30° - \alpha)$. We obtain from $\triangle MOP$ that $\angle BOK = \angle MOP = 90° - \alpha$. We find from $\triangle BOK$ that $BK = BO \sin \angle BOK = a \sin (90° - \alpha) = a \cos \alpha$. Since $\angle POE = 90° + \alpha$ (as an exterior angle for $\triangle MOP$) and $\angle POC = 60°$, we have: $\angle COE = \angle POE - \angle POC = (90° + \alpha) - 60° = 30° + \alpha$. We find from $\triangle COE$ that $CE = CO \sin \angle COE = a \sin (30° + \alpha)$. We have: $AD^2 + BK^2 + CE^2 = a^2 \sin^2 (30° - \alpha) + a^2 \cos^2 \alpha + a^2 \sin^2 (30° + \alpha)$

$$= a^2 \left(\frac{1 - \cos (60° - 2\alpha)}{2} + \cos^2 \alpha + \frac{1 - \cos (60° + 2\alpha)}{2} \right)$$

$$= a^2 \left(1 - \frac{\cos (60° + 2\alpha) + \cos (60° - 2\alpha)}{2} + \cos^2 \alpha \right)$$

$$= a^2 \left(1 - \cos 60° \cos 2\alpha + \frac{1 + \cos 2\alpha}{2} \right)$$

$$= a^2 \left(1 - \frac{1}{2} \cos 2\alpha + \frac{1}{2} + \frac{1}{2} \cos 2\alpha \right) = \frac{3}{2} a^2.$$

Thus, for any α we have: $AD^2 + BK^2 + CE^2 = \frac{3}{2} a^2$.

Example 10. Find the relationship between the sides a, b, and c of the triangle ABC if it is known that the median AM, the altitude BH, and the angle bisector CD intersect at a common point (Fig. 43).

Solution. By Ceva's theorem (Theorem 10), we have: $\frac{AD}{BD} \times$

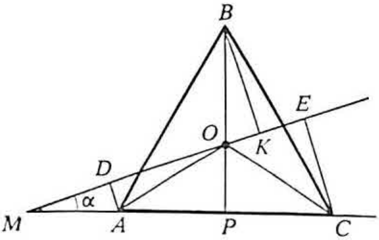


Fig. 42

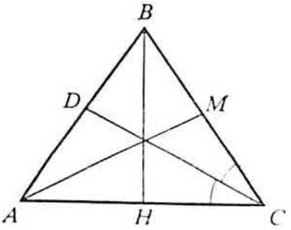


Fig. 43

$\frac{BM}{CM}\cdot\frac{CH}{AH}=1$. Since AM is a median, $BM=CM$ and $\frac{BM}{CM}=1$. Since CD is an angle bisector, $\frac{AD}{BD}=\frac{b}{a}$ (Theorem 5). Thus, the given relationship takes the form: $\frac{b}{a}\cdot\frac{CH}{AH}=1$, i.e. $\frac{CH}{AH}=\frac{a}{b}$.

We set $CH=at$ and $AH=bt$. Then, on the one hand, $at+bt=b$, i.e. $t=\frac{b}{a+b}$. On the other hand, using BH as a reference element, we find from $\triangle ABH$ that $BH^2=c^2-b^2t^2$ and from $\triangle BHC$ that $BH^2=a^2-a^2t^2$. Hence, $c^2-b^2t^2=a^2-a^2t^2$, whence $t^2=\frac{a^2-c^2}{a^2-b^2}$.

Substituting the value of $t=\frac{b}{a+b}$ into the last equality, we get the desired relationship between the sides a, b, and c: $\frac{b^2}{(a+b)^2}=\frac{a^2-c^2}{a^2-b^2}$, i.e. $\frac{b^2}{a+b}=\frac{a^2-c^2}{a-b}$.

PROBLEMS TO BE SOLVED WITHOUT ASSISTANCE

I. Right Triangles

1. Prove that in a right triangle, the bisector of the right angle halves the angle between the median and altitude drawn from the same vertex.

2. The median drawn to the hypotenuse of a right triangle divides the right angle in the ratio 1:2 and is equal to m. Find the sides of the triangle.

3. A point taken on the hypotenuse of a right triangle and equidistant from its legs divides the hypotenuse into two segments, 30 cm and 40 cm long. Find the legs.

4. Find the bisectors of acute angles of a right triangle whose legs are 18 cm and 24 cm long.

5. Find the bisector of the right angle of a right triangle with legs a and b.

6. Drawn from the right angle of a right triangle is an angle bisector which divides the hypotenuse into line segments, m and n. Find the altitude drawn to the hypotenuse.

7. In a right triangle, the medians drawn to the legs are equal to $\sqrt{52}$ cm and $\sqrt{73}$ cm. Find the hypotenuse.

8. The perimeter of a right triangle is equal to 60 cm, and the altitude drawn to the hypotenuse to 12 cm. Find the sides of the triangle.

9. In a right triangle ABC, an angle bisector CK and a median CM are drawn from the vertex C of the right angle. Find the legs if $CM = m$ and $KM = n$.

10. In a right triangle, find the angle between the median and angle bisector drawn from the vertex of the acute angle equal to α.

11. The bisectors of the acute angles AD and BK are drawn in a right triangle ABC. Find the angles of the triangle if it is known that $AB^2 = AD \cdot BK$.

12. Prove that if the altitude and median drawn from the same vertex of a nonisosceles triangle lie inside the triangle and form equal angles with its sides, then this is a right triangle.

II. Isosceles Triangles

13. Prove that if in a triangle, the ratio of the tangents of two angles is equal to the ratio of the squares of the sines of these angles, then the triangle is either an isosceles or a right one.

14. Prove that if the relationship $\frac{a}{\cos A} = \frac{b}{\cos B}$ is fulfilled in a triangle, then the triangle is isosceles.

15. The base of an isosceles triangle is equal to $4\sqrt{2}$ cm, and the median drawn to a lateral side is 5 cm long. Find the side.

16. The lateral side of an isosceles triangle is equal to 4 cm, and the median drawn to this side is equal to 3 cm. Find the base of the triangle.

17. The base of an isosceles triangle is equal to 12 cm, and the lateral side to 18 cm. The altitudes are drawn to the lateral sides. Compute the length of the line segment for which the feet of the altitudes serve as end points.

18. The base of an isosceles triangle is equal to 12 cm, and the lateral side to 18 cm. Angle bisectors are drawn to the lateral sides. Compute the length of the line segment for which the feet of the bisectors serve as end points.

19. The sum of two unequal altitudes of an isosceles triangle is equal to l, and the vertex angle is α. Find the lateral side.

20. A point M is taken on the altitude BH of an isosceles triangle ABC ($AB = BC$) so that the angles AMB, BMC, and AMC are congruent. In what ratio (reckoning from the vertex) does the point M divide the altitude if the base angle equals α?

21. The base angle of an isosceles triangle is equal to α. Find the ratio of the base to the median drawn to the lateral side.

22. Find the angles of an isosceles triangle if it is known that the orthocentre bisects the altitude drawn to the base of the triangle.

23. Straight lines l_1, l_2, and l_3 are parallel, l_2 lying between l_1 and l_3 at a distance p and q from them, respectively. Find the side of the regular triangle whose vertices lie on the given lines (one on each line).

24. A point D is taken on the side AB and a point E on the side BC of an isosceles triangle ABC ($AB = BC$) so that $BD = CE$. Prove that the set of midpoints of all the line segments, DE, coincides with the midline of the triangle ABC.

25. In an isosceles triangle, the angle at the vertex is equal to 36°, and the base is equal to a. Find the lateral sides of the triangle.

26. A point D is taken on the side BC of an isosceles triangle ABC ($AB = BC$) so that $BD:DC = 1:4$. Find $BM:ME$, where BE is the altitude of the triangle, and M the point of intersection of BE and AD.

27. The base of an isosceles triangle is equal to a, and the angle at the vertex to 2α. Find the length of the angle bisector drawn to the lateral side.

28. Drawn through a vertex of a regular triangle is a ray dividing the base in the ratio 2:1. What are the angles formed by this ray and the lateral sides of the triangle?

29. The base angle of an isosceles triangle is equal to $\arctan \frac{3}{4}$. Find the angle between a median and an angle bisector drawn to a lateral side.

30. Find the angle at the vertex of an isosceles triangle if the median drawn to the lateral side forms an angle of $\arcsin \frac{3}{5}$ with the base.

31. The angle B of an isosceles triangle ABC is equal to 110°. Inside the triangle, a point M is taken so that $\angle MAC = 30°$ and $\angle MCA = 25°$. Find the angle BMC.

III. Arbitrary Triangles

32. Prove that if two sides and the altitude of one acute triangle are respectively equal to two sides and the altitude of another acute triangle, then such triangles are congruent (consider two cases).

33. Prove that if two sides and a median of one triangle are respectively equal to two sides and a median of the other triangle, then such triangles are congruent (consider two cases).

34. Prove that in any triangle, an angle bisector either coincides with the median and altitude drawn from the same vertex or lies between them.

35. Prove that in any triangle, the sum of medians is more than $\frac{3}{4}$ of its perimeter, but less than the whole perimeter.

36. Drawn through the centroid of a regular triangle ABC is a straight line l intersecting the sides AB and BC. Prove that the sum of the distances from A and C to l is equal to the distance from B to l.

37. The angle B of a triangle ABC is equal to 115°. Drawn from the midpoint of the side AC is a perpendicular to AC to intersect the side BC at the point D. The line segment AD divides the angle A in the ratio 5:3, reckoning from the side AC. Find the angles A and C of the triangle ABC.

38. The bisector of an angle of a triangle divides the opposite side into two line segments, 2 cm and 4 cm long, and the altitude drawn to the same side is equal to $\sqrt{15}$ cm. Find the sides of the triangle and determine its kind.

39. Find the ratio of the sum of the squares of the medians of a triangle to the sum of the squares of its sides.

40. Determine the kind of a triangle if it is known that its medians are related by the equality $m_a^2 + m_b^2 = 5m_c^2$.

41. Two sides of a triangle are equal to a and b, and the medians drawn to these sides are mutually perpendicular. Find the third side of the triangle.

42. An angle bisector AD is drawn in a triangle ABC. Find BC if it is known that $AC = b$, $AB = c$, and $AD = BD$.

43. In a triangle ABC, $BC = 12$ cm, $AC = 8$ cm, and the angle A is twice the angle B. Find AB.

44. The altitude of a triangle is equal to 6 cm and divides the vertex angle in the ratio 2:1 and the base of the triangle into the segments the smaller of which is equal to 3 cm. Find the sides of the triangle.

45. The altitude of a triangle divides the vertex angle in the ratio 2:1 and the base of the triangle into the segments whose ratio is equal to k $(k > 1)$. Find the larger angle at the base of the triangle.

46. In an acute triangle ABC, the acute angle between the altitudes AD and CE is equal to α. Find AC if $AD = a$ and $CE = b$.

47. The base of a triangle is equal to 4. The median drawn to the base equals $\sqrt{6} - \sqrt{2}$, and one of the base angles measures 15°. Find the acute angle between the median and base.

48. Taking advantage of Ceva's theorem, prove that:

(a) the medians of a triangle intersect at a common point;

(b) the bisectors of the angles of a triangle intersect at a common point;
(c) the altitudes of a triangle intersect at a common point.

49. The straight line DE is parallel to the base AC of a triangle ABC, the point D lying on the side AB and the point E on the side BC. Prove that AE, CD, and the median BM intersect at a common point.

50. Prove that if the lengths of the sides of a triangle form an arithmetic progression, then the centre of the circle inscribed in this triangle and the centroid of the triangle lie on a straight line parallel to the side having a medium length.

51. In a triangle ABC, AD is the altitude, and the point H the orthocentre. Prove that $DC \cdot DB = AD \cdot DH$.

52. In a triangle ABC, the angle A is equal to 30° and the angle B to 50°. Prove that the sides of the triangle are related as follows: $c^2 = b(a + b)$.

53. Prove that in any triangle, the difference between the sum of the squares of the lengths of any two of its sides and the product of the lengths of these sides multiplied by the included angle is a constant (for the given triangle).

54. Prove that in any triangle, the orthocentre, the centroid, and the centre of the circumscribed circle lie on a common straight line (*Euler's line*).

55. In triangles ABC and $A'B'C'$, the angles B and B' are congruent and the sum of the angles A and A' is equal to 180°. Prove that the sides of these triangles are connected by the relationship $aa' = bb' + cc'$.

56. In a triangle ABC, the angles A, B, and C are in the ratios 4:2:1. Prove that the sides of the triangle are related by the equality $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$.

57. The altitude, median, and angle bisector drawn from one and the same vertex of a triangle divide the angle at this vertex into four equal parts. Find the angles of the triangle.

58. In a triangle ABC, CD is the altitude. Find the relationship between the angles A and B if it is known that $CD^2 = AD \cdot DB$.

59. In a triangle ABC, the angle A is equal to α, the angle B to β, and the median BD intersects the angle bisector CE at the point K. Find $CK:KE$.

IV. Parallelograms

60. Prove that if in a quadrilateral, the diagonals serve as bisectors of its angles, then such a quadrilateral is a rhombus.

61. In a parallelogram with sides a and b $(a > b)$, the bisectors of the interior angles are drawn. Determine the kind of the quadrilateral formed by the intersection of the angle bisectors and find the lengths of its diagonals.

62. The altitude of a rhombus divides its side into two line segments, m and n. Find the diagonals of the rhombus.

63. The perpendicular dropped from a vertex of a parallelogram to a diagonal divides this diagonal into two segments, 6 cm and 15 cm long. Find the sides and diagonals of the parallelogram if it is known that the difference between the sides is equal to 7 cm.

64. Two altitudes of a parallelogram drawn from the vertex of an obtuse angle are equal to p and q, respectively, and the angle between them is equal to α. Find the larger diagonal of the parallelogram.

65. The diagonal of a rectangle divides its angle in the ratio $m:n$. Find the ratio of the perimeter of the rectangle to its diagonal.

66. The acute angle of a parallelogram is equal to α, and the sides to a and b. Find the tangents of the acute angles formed by the larger diagonal and the sides of the parallelogram.

67. Find the acute angle of a rhombus $ABCD$ if the straight line drawn through the vertex A divides the angle BAD in the ratio 1:3, and the side BC in the ratio 3:5.

68. The ratio of the perimeter of a rhombus to the sum of its diagonals is equal to k. Find the angles of the rhombus.

69. The diagonals of a parallelogram are proportional to its nonparallel sides. Prove that the angles between the diagonals are equal to the angles of the parallelogram.

70. In a rectangle $ABCD$, the base AD is divided into three equal parts by the points M and P. Prove that the sum of the angles AMB, APB, and ADB is equal to 90° if it is known that $AD = 3AB$.

71. The sides of a parallelogram are equal to a and b $(a < b)$. The smaller diagonal forms an obtuse angle with the smaller side, and an angle α with the larger side. Find the larger diagonal of the parallelogram.

72. The sides of a parallelogram are to each other as $p:q$, and the diagonals as $m:n$. Find the angles of the parallelogram.

73. The ratio of the perimeter of a parallelogram to its larger diagonal is equal to k. Find the angles of the parallelogram if it is known that the larger diagonal divides the angle of the parallelogram in the ratio 1:2.

V. Trapezoids

74. Prove that if the sides of one trapezoid are respectively equal to the sides of the other trapezoid, then the trapezoids are congruent.

75. Prove the following theorems: in order for a trapezoid to be isosceles, it is necessary and sufficient that: (a) the base angles be equal; (b) the diagonals be equal.

76. Prove that the bisectors of the angles adjacent to the lateral side of a trapezoid intersect at right angles, and the point of their intersection lies on the median of the trapezoid.

77. The sum of the base angles of a trapezoid is equal to 90°. Prove that the line segment joining the midpoints of the bases is equal to the half-difference between the bases.

78. The diagonals of a trapezoid are equal and mutually perpendicular, the altitude being equal to 15 cm. Find the length of the median of the trapezoid.

79. One of the bases of a trapezoid is equal to 24 cm, and the distance between the midpoints of the diagonals to 4 cm. Find the other base.

80. One of the angles of a trapezoid is equal to 30°, the lateral sides being mutually perpendicular. Find the smaller lateral side of the trapezoid if its median is equal to 10 cm, and one of the bases to 8 cm.

81. In a right trapezoid, the bases and the smaller lateral side are equal to a, b, and c. Find the distances from the point of intersection of the diagonals to the bases and the smaller lateral side.

82. The point of intersection of the bisectors of obtuse angles at a base of a trapezoid is found on the other base, their lengths being 13 cm and 15 cm. Find the sides of the trapezoid if its altitude is equal to 12 cm.

83. The altitude of an isosceles trapezoid is h, the acute angle between the diagonals is equal to 2α. Find the length of the median of the trapezoid.

84. In a trapezoid $ABCD$, A and B are right angles, $AB = 5$ cm, $BC = 1$ cm, and $AD = 4$ cm. A point M is taken on the side AB so that the angle AMD is twice the angle BMC. Find the ratio $AM:MB$.

85. The angle at the vertex A of a trapezoid $ABCD$ is equal to α, and the lateral side AB is twice the length of its smaller base BC. Find the angle BAC.

86. The larger base of a trapezoid is equal to a, the lateral sides are b and c $(b < c)$, and the angles at the larger base are to each other as 2:1. Find the smaller base.

87. The diagonals AC and BD of an isosceles trapezoid $ABCD$ $(AD \parallel BC)$ intersect at the point O, the angle AOD being equal to 60°. Prove that the points K, M, and P serving respectively as the midpoints of the line segments AO, BO, and CD are the vertices of a regular triangle.

88. Prove that the sum of the squares of the diagonals of a trapezoid is equal to twice the product of its bases plus the sum of the squares of the lateral sides.

89. Prove that the straight line passing through the point of intersection of the extended lateral sides of a trapezoid and the point of intersection of its diagonals bisects the bases of the trapezoid.

90. In a trapezoid with the bases a and b, drawn through the point of intersection of the diagonals is a straight line parallel to the bases. Find the length of the segment of this line enclosed between the lateral sides of the trapezoid.

91. In a trapezoid $ABCD$, each of the bases AD and BC is extended in both directions. The bisectors of the exterior angles A and B intersect at the point K, and the bisectors of the exterior angles C and D intersect at the point E. Find the perimeter of the trapezoid if $KE = 20$ cm.

92. Through the point O of intersection of the diagonals of an isosceles trapezoid $ABCD$ $(AD \parallel BC)$ having mutually perpendicular diagonals, a straight line MK is drawn perpendicular to the side CD (the point M lies on AB, and the point K on CD). Find MK if $AD = 40$ cm and $BC = 30$ cm.

VI. Miscellaneous Problems

93. In a quadrilateral $ABCD$, the points P, K, E, and M are the midpoints of the sides AB, BC, CD, and DA, respectively. Prove that a quadrilateral $PKEM$ is a parallelogram.

94. Constructed on the legs AC and BC of a right triangle ABC are squares $ADKC$ and $CEMB$. The perpendiculars DH and MP are dropped from the points D and M to the extension of the hypotenuse AB. Prove that $DH + MP = AB$.

95. Squares are constructed on the sides of a parallelogram (outside it). Their centres are joined in succession. Prove that the quadrilateral thus obtained is a square.

96. Inscribed in a right triangle with legs a and b is a square having a common right angle with the triangle. Find the perimeter of the square.

97. In a right triangle, a rhombus is inscribed so that all of its vertices lie on the sides of the triangle, and the angle, equal to 60°, is a common angle for the triangle and rhombus. Find the sides of the triangle if the rhombus is 6 cm on a side.

98. A rhombus is inscribed in a triangle so that they have one angle in common, and the opposite vertex of the rhombus divides the side of the triangle into two segments whose ratio is 2:3. Find the sides of the triangle including the common angle of the triangle and rhombus if the diagonals of the rhombus are equal to m and n.

99. In a triangle with lateral sides of 9 cm and 15 cm, a parallelogram is inscribed so that one of its sides, equal to 6 cm, lies on the base of the triangle, and the diagonals are parallel to the lateral sides of the triangle. Find the other sides of the parallelogram and the base of the triangle.

100. Inscribed in a square $ABCD$ is an isosceles triangle AKM so that the point K lies on the side BC, the point M on CD, and $AM = AK$. Find the angle MAD if it is known that $\tan \angle AKM = 3$.

101. Inscribed in a regular triangle ABC is a regular triangle DEK so that the point D lies on the side BC, the point E on AC, and the point K on AB. Find $AB{:}DE$ if it is known that $\angle DEC = \alpha$.

102. Prove that the line segments joining the midpoints of the opposite sides of a convex quadrilateral and the line segment connecting the midpoints of its diagonals intersect at a common point and are bisected by this point.

103. Prove that in a convex quadrilateral, the midpoints of its diagonals and the midpoints of the line segments joining the midpoints of its opposite sides lie on one straight line.

104. Prove that if in a quadrilateral, the sums of the squares of its opposite sides are equal, then its diagonals are mutually perpendicular.

105. Prove that if the line segment joining the midpoints of two opposite sides of a convex quadrilateral is equal to half the sum of two other sides, then this quadrilateral is a trapezoid.

106. The bases of two regular triangles with sides a and $3a$ lie on one and the same straight line. The triangles are situated on different sides of the line, and the distance between the nearest end points of their bases is equal to $2a$. Find the distance between the vertices of the triangles not lying on the given line.

107. In a quadrilateral $ABCD$, $\angle A = \angle D = 60°$, $AB = \sqrt{3}$, $BC = 3$, and $CD = 2\sqrt{3}$. Find the angles B and C.

108. The diagonals of a convex quadrilateral $ABCD$ intersect at the point O at right angles so that $AO = 8$ cm, $BO = CO = 1$ cm, and $DO = 7$ cm. When extended, the sides AB and CD intersect at the point M. Find the angle AMD.

109. In a quadrilateral $ABCD$, B is a right angle, and $AB:BD = 2:4\sqrt{2}$. When extended, the sides BC and AD intersect at the point M. Find the angle DMC if $\angle ABD = 45°$.

110. Inscribed in a rectangle $ABCD$ is a triangle AEK so that the point E lies on the side BC, and the point K on CD. Find the angle tan $\angle EAK$ if $\frac{AB}{BC} = \frac{BE}{CE} = \frac{CK}{DK} = m$.

SEC. 3. CIRCLES

Example 1. Prove that if a and b are the legs, c the hypotenuse of a right triangle, and r is the radius of the inscribed circle, then $r = \frac{a+b-c}{2}$.

Solution. Let us carry out all necessary auxiliary constructions: from the centre O of the inscribed circle we draw the radii OD, OE, and OF to the points of tangency. Then $OD \perp BC$, $OE \perp AC$, and $OF \perp AB$ (Fig. 44). Since $ODCE$ is a square (all the angles are right ones, and $OE = OD$), we have: $CE = CD = r$, $BD = a - r$, and $AE = b - r$. But $BD = BF$ and $AE = AF$ (Theorem 12b), hence, $BF = a - r$ and $AF = b - r$. Since $AB = BF + AF$, that is, $c = (a - r) + (b - r)$, we find that $r = \frac{a+b-c}{2}$.

Remark. The formula thus obtained is frequently used for solving problems dealing with right triangles.

Example 2. A tangent is drawn to a circle inscribed in a triangle having a perimeter of 18 cm, the tangent being parallel to the base of the triangle. The length of the line segment of the tangent enclosed between the lateral sides of the triangle is equal to 2 cm. Find the base of the triangle.

Solution. Let M, P, and N denote the points of tangency (Fig. 45). Then $AM = AN$, $CN = CP$, and $BP = BM$ (Theorem 12b). We set $AN = AM = x$, $CN = CP = y$, and $BP = BM = z$. Then the perimeter of the triangle ABC is equal to $2x + 2y + 2z$, and, therefore, $x + y + z = 9$.

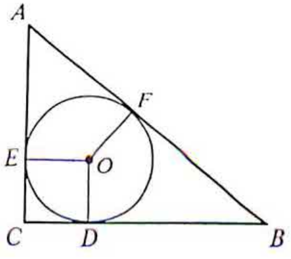


Fig. 44

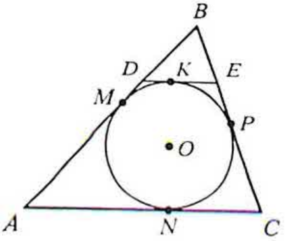


Fig. 45

We draw the tangent DE parallel to AC. Then the triangles ABC and DBE are similar, and, therefore, the ratio of their sides is equal to that of their perimeters: $\frac{DE}{AC}=\frac{P_{DBE}}{P_{ABC}}$, that is,

$$\frac{2}{x+y}=\frac{P_{DBE}}{18}, \tag{1}$$

where $P_{DBE}=BD+BE+DE=BD+BE+(DK+KE)=BD+BE+(DM+EP)$. (Here we have taken advantage of the fact that $DM=DK$ and $KE=EP$.) Hence, $P_{DBE}=(BD+DM)+(BE+EP)=BM+BP=2z$, and then Equality (1) can be rewritten as follows: $\frac{2}{x+y}=\frac{2z}{18}$. Thus, we have obtained the system of equations: $\begin{cases} x+y+z=9, \\ \frac{2}{x+y}=\frac{z}{9}. \end{cases}$ Setting $x+y=b$, we get: $\begin{cases} b+z=9, \\ bz=18, \end{cases}$ whence we find that either $b=3$ cm or $b=6$ cm.

Example 3. Drawn through the point A of a common chord AB of two circles is a straight line intersecting the first circle at the point C, and the second circle at the point D. The tangent to the first circle at the point C and the tangent to the second circle at the point D intersect at the point M. Prove that the points M, C, B, and D lie on one and the same circle (Fig. 46).

Solution. It is sufficient to prove that $\angle CMD+\angle CBD=180^\circ$ (Theorem 15a). Then $\angle CBA=\frac{1}{2}\cup AC$ (as an inscribed angle). In the same way $\angle MCA=\frac{1}{2}\cup AC$ (as an angle between a tangent and a chord,

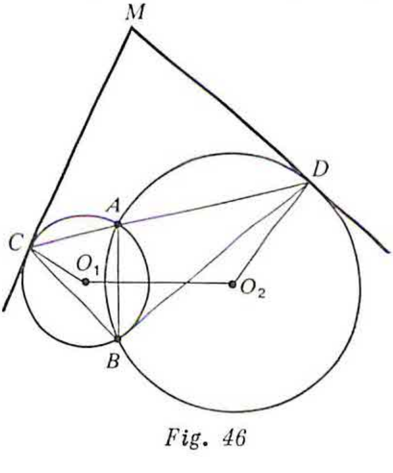


Fig. 46

Theorem 13). Hence, $\angle CBA = \angle MCA$. It is proved in a similar way that $\angle ABD = \angle ADM$. We conclude from the triangle MCD that $\angle CMD + \angle MCD + \angle MDC = 180^\circ$. But $\angle MCD + \angle MDC = \angle CBA + \angle ABD = \angle CBD$.
Hence, $\angle CMD + \angle CBD = 180^\circ$, which was required to be proved.

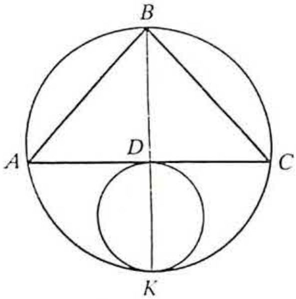


Fig. 47

Example 4. Inscribed in a circle is an isosceles triangle ABC whose base $AC = b$ and the base angle is α. A second circle touches the first circle and the base of the triangle at its midpoint D, and is situated outside the triangle. Find the radius of the second circle (Fig. 47).

Solution. Let us take advantage of the fact that $AD \cdot DC = BD \cdot DK$ (Theorem 16a). Since $AD = DC = \frac{b}{2}$, $BD = \frac{b}{2}\tan\alpha$, and $DK = 2r$, we get: $\frac{b^2}{4} = \frac{b}{2}\tan\alpha \cdot 2r$, whence $r = \frac{b}{4}\cot\alpha$.

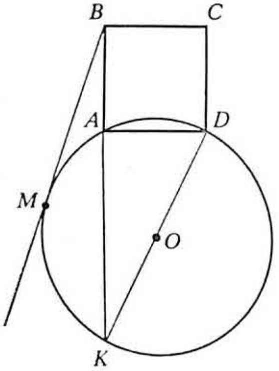


Fig. 48

Example 5. A circle of radius R passes through two adjacent vertices of a square. A tangent to the circle drawn from a third vertex of the square is twice the length of the side of the square. Find the side of the square.

Solution. Let us denote $AB = x$ and $BM = 2x$ (Fig. 48). We extend the line segment AB to intersect the circle at the point K. Then $BK \cdot AB = BM^2$ (Theorem 16c), that is, $BK \cdot x = 4x^2$, whence we find that $BK = 4x$, and, hence, $AK = 3x$. The angle KAD is equal to 90°, and, consequently, KD is a diameter. From a right triangle ADK we find that $AD^2 + AK^2 = KD^2$, that is, $x^2 + 9x^2 = 4R^2$, whence $x = \frac{R\sqrt{10}}{5}$.

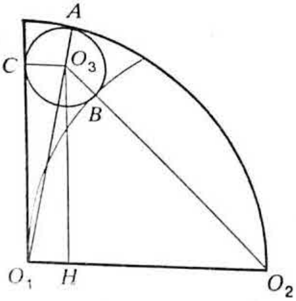


Fig. 49

Example 6. Given a right sector of a circle. A circle of the same radius with centre at the end point of the arc of the sector separates the sector into two curvilinear triangles. A circle is inscribed in the smaller of these triangles. Find the ratio of the radii of the inscribed circle and the sector.

Solution. We carry out all necessary auxiliary constructions which are usually made when dealing with two circles, internally or externally tangent to each other, or with a straight line tangent to a circle: O_2O_3 is the line of centres, B the point of tangency, O_1O_3 the line of centres, A the point of tangency, O_3C is perpendicular to O_1C, and C the point of tangency (Fig. 49). Let us denote $O_1O_2 = R$ (an auxiliary parameter) and express the radius r of the inscribed circle in terms of R.

Consider the triangle $O_1O_2O_3$. We have: $O_1O_2 = R$, $O_1O_3 = R - r$, and $O_2O_3 = R + r$. Draw the altitude O_3H. Then $O_1H = O_3C = r$ and $O_2H = R - r$. We use O_3H as a reference element. We get from the triangle O_1O_3H: $O_3H^2 = O_1O_3^2 - O_1H^2 = (R-r)^2 - r^2$. We obtain from the triangle O_3HO_2: $O_3H^2 = O_2O_3^2 - O_2H^2 = (R+r)^2 - (R-r)^2$.

Thus, $(R-r)^2 - r^2 = (R+r)^2 - (R-r)^2$, whence we find that $r = \frac{R}{6}$. Hence, $\frac{r}{R} = \frac{1}{6}$.

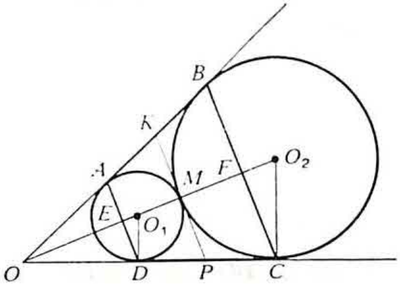


Fig. 50

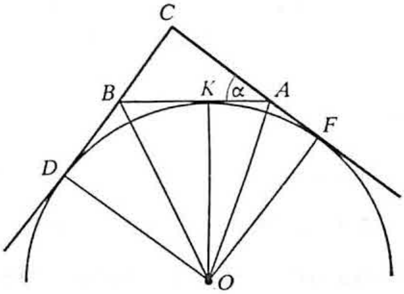


Fig. 51

Example 7. Two circles of radii r and R touch each other externally, AB and CD being their common external tangents. Prove that a circle can be inscribed in the quadrilateral $ABCD$ and find the radius of this circle (Fig. 50).

Solution. We carry out all necessary auxiliary constructions. Extend the tangents to intersect each other at the point O, draw the line of centres OO_1O_2, draw the radii O_1D and O_2C to the points of tangency: $O_1D \perp CD$ and $O_2C \perp CD$.

Since the line of centres is the axis of symmetry of a figure, the points A and D are symmetric about OO_2, and the points B and C are symmetric about OO_2. This means that $ABCD$ is an isosceles trapezoid.

In order for a circle to be inscribed in the trapezoid $ABCD$, it is necessary and sufficient that the equality $AD + BC = AB + CD$ (Theorem 16b) or, since $AB = CD$, the equality $AB = \frac{AD + BC}{2}$

be fulfilled. Hence, it suffices to prove that the line segment AB is equal to the median of the trapezoid.

We draw an internal common tangent KP. Then $AK = KM$, $BK = KM$, $DP = PM$, and $CP = PM$ (Theorem 12b). which means that KP is the median of the trapezoid $ABCD$ and $KP = AB$. Thus, a circle can be inscribed in the trapezoid, EF being its diameter. We set $O_1E = x$ and $O_2F = y$. Then from the equality $MF = ME$ (the median KP bisects the line segment EF) we conclude that $R - y = r + x$. We get from the similarity of triangles O_1DE and O_2CF: $\frac{O_1E}{O_2F} = \frac{O_1D}{O_2C}$, that is, $\frac{x}{y} = \frac{r}{R}$. We find from the system of equations $\begin{cases} R - y = r + x, \\ \frac{x}{y} = \frac{r}{R} \end{cases}$ that $y = \frac{R^2 - rR}{R - r}$, and then the radius of the inscribed circle is equal to $R - y = \frac{2Rr}{R+r}$.

Example 8. An acute angle of a right triangle equals α. Find the hypotenuse of this triangle if the radius of the circle touching the hypotenuse and the extended legs is equal to R (Fig. 51).

Solution. Since $AB = AK + BK$, the problem is reduced to computing the line segments AK (from $\triangle AOK$) and BK (from $\triangle OBK$). Consider the triangle AOK. Since $\angle KOF = \angle BAC = \alpha$ (as angles with mutually perpendicular sides), we have: $\angle KOA = \frac{\alpha}{2}$ ($\triangle KOA = \triangle AOF$). Hence, $AK = OK \tan \frac{\alpha}{2} = R \tan \frac{\alpha}{2}$.

Consider the triangle BOK. We have: $\angle BOK = \frac{1}{2} \angle DOK = \frac{1}{2}(90° - \angle KOF) = 45° - \frac{\alpha}{2}$. (Here we have taken advantage of the fact that in the quadrilateral $ODCF$ three angles (D, C, and F) are right, and, hence, the fourth angle, that is, the angle DOF, is also right.) Then $BK = OK \tan \angle BOK = R \tan \left(45° - \frac{\alpha}{2}\right)$,

$$AB = AK + BK = R \tan \frac{\alpha}{2} + R \tan \left(45° - \frac{\alpha}{2}\right) = R\,\frac{\sin\left(\frac{\alpha}{2} + 45° - \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}\cos\left(45° - \frac{\alpha}{2}\right)} = \frac{R\sqrt{2}.}{2\cos\frac{\alpha}{2}\cos\left(45° - \frac{\alpha}{2}\right)}.$$

(Here we have used the formula $\tan\alpha + \tan\beta = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$.)

Example 9. Given an acute triangle ABC with angles $\angle A = \alpha$, $\angle B = \beta$, and $\angle C = \gamma$ (Fig. 52). In what ratio is the altitude drawn from the vertex A divided by the orthocentre?

Solution. We circumscribe a circle about the triangle ABC and denote its radius by R (an auxiliary parameter). We then draw OP

perpendicular to BC and take advantage of the fact that $AH = 2OP$ (see Example 8, Sec. 2), where H is the orthocentre.

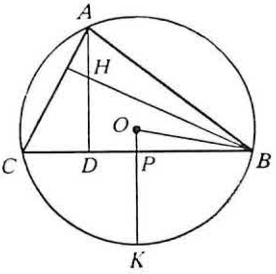


Fig. 52

Consider the triangle OPB. Since the angle KOB is measured by the arc BK, $\cup BK = \frac{1}{2} \cup BC$, and the angle CAB is measured by half the arc BC, we have: $\angle KOB = \angle CAB = \alpha$. Then $OP = R \cos \alpha$, and, therefore, $AH = 2R \cos \alpha$. By the law of sines, $\frac{AC}{\sin \angle ABC} = 2R$ (Theorem 8), hence, $AC = 2R \sin \beta$, and then we find from $\triangle ACD$ that $AD = AC \sin \angle ACB = 2R \sin \beta \sin \gamma$. Then $AH = 2R \cos \alpha$ and $HD = AD - AH = 2R \sin \beta \sin \gamma - 2R \cos \alpha = 2R (\sin \beta \sin \gamma - \cos (180° - (\beta + \gamma))) = 2R (\sin \beta \sin \gamma + \cos (\beta + \gamma)) = 2R (\sin \beta \sin \gamma \times \cos \beta \cos \gamma - \sin \beta \times \sin \gamma) = 2R \cos \beta \cos \gamma$.

Thus, $\frac{AH}{HD} = \frac{2R \cos \alpha}{2R \cos \beta \cos \gamma} = \frac{\cos \alpha}{\cos \beta \cos \gamma}$.

Example 10. Prove that if the altitude and median drawn from one and the same vertex of a nonisosceles triangle lie inside the triangle and form equal angles with its lateral sides, then this is a right triangle.

Solution. We circumscribe a circle about the given triangle ABC and extend the altitude BD and the median BM to intersect the circle at the points E and K, respectively (Fig. 53). Since $\angle ABE = \angle KBC$, we have: $\cup AE = \cup KC$, and, therefore, the chords AC and EK enclosing the equal arcs AE and KC are parallel to each other. But $\angle BDM = 90°$, hence, $\angle BEK = 90°$, and then BK is a diameter of the circle.

Since the centre of the circumscribed circle lies, on the one hand, on the diameter BK, and, on the other hand, on the perpendicular to AC erected from the point M, this centre is the point M itself. Hence, AC is a diameter of the circle, and, therefore, $\angle ABC = 90°$.

Example 11. In an acute triangle ABC, AD and CM are its altitudes, the perimeter of the triangle ABC is equal to 15 cm, the perim-

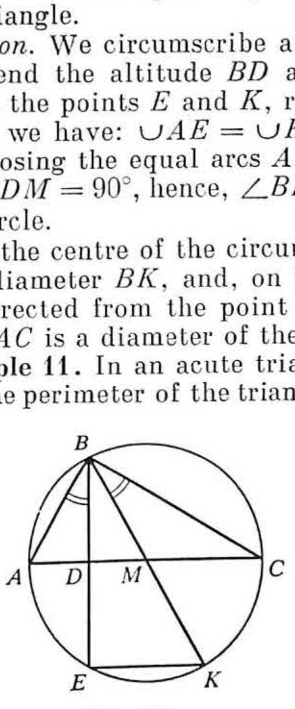


Fig. 53

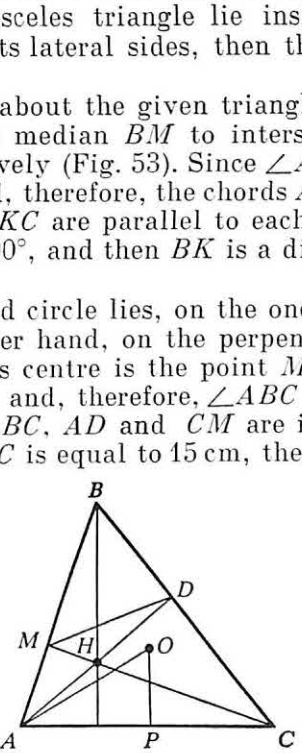


Fig. 54

eter of the triangle MBD is equal to 9 cm, and the radius of the circle circumscribed about the triangle MBD is equal to 1.8 cm. Find the length of AC (Fig. 54).

Solution. First of all let us prove that the triangles ABC and MBD are similar. Indeed, the right triangles ABD and MBC, having a common acute angle B, are similar, and, therefore, $\frac{AB}{BC}=\frac{BD}{BM}$. But then, in the triangles ABC and MBD with a common angle B, the sides including this angle are proportional, that is, the triangles are similar.

Let us take advantage of the fact that in similar triangles, the ratio of the perimeters and the ratio of the radii of the circumscribed circles are equal to the ratio of similitude. By the hypothesis, $P_{ABC} = 15$ cm and $P_{MBD} = 9$ cm. Hence, the ratio of similitude is equal to $\frac{5}{3}$. Since the radius of the circle circumscribed about the triangle MBD is equal to 1.8 cm, we get that the radius of the circle circumscribed about the triangle ABC is equal to $1.8\cdot\frac{5}{3}=3$ cm.

Let O be the centre of the circle circumscribed about the triangle ABC, and OP perpendicular to AC. Then $BH = 2OP$ (see Example 8, Sec. 2). But BH is a diameter of the circle circumscribed about the triangle MBD (since the angle BDH based on it is equal to 90°), hence, $BH = 3.6$ cm, and, therefore, $OP = 1.8$ cm.

Now, in the right triangle AOP, two sides are known, that is, $AO = 3$ cm (the radius of the circumscribed circle) and $OP =$ 1.8 cm. Then $AP=\sqrt{9-\left(\frac{9}{5}\right)^2}=\frac{12}{5}$ cm, and, consequently, $AC = 4.8$ cm.

PROBLEMS TO BE SOLVED WITHOUT ASSISTANCE

I. Circles

111. Two circles touch each other externally at a point A, BC being their common external tangent. Prove that $\angle BAC = 90°$.

112. Two circles intersect at points A and B. The points A and B lie on different sides of the straight line l which intersects the circles at points C, D, E, and M. Prove that the sum of the angles DBE and CAM is equal to 180°.

113. Two circles intersect at points A and B. Straight lines l_1 and l_2 are parallel, l_1 passing through the point A and intersecting the circles at points E and K, and l_2 passing through the point B and intersecting the circles at points M and P. Prove that the quadrilateral $EKMP$ is a parallelogram.

114. Drawn from a point M to a circle with centre at O are two tangents MA and MB. A straight line l touches the circle at a point C and intersects MA and MB at points D and E, respectively. Prove that: (a) the perimeter of the triangle MDE is independent of the choice of the point C; (b) the angle DOE is independent of the choice of the point C.

115. Points A, B, C, and D divide a circle into four parts whose ratios are 1:3:5:6. Find the angles between the tangents to the circle drawn at the points A, B, C, and D.

116. Two equal circles touch externally each other and the third circle whose radius is equal to 8 cm. The line segment joining the points of tangency of two equal circles and the third circle is equal to 12 cm. Find the radius of the equal circles.

117. A common chord of two intersecting circles is equal to a and serves as a side of a regular hexagon inscribed in one circle and as a side of a square inscribed in the other circle. Find the distance between the centres of the circles.

118. Two circles of radii r and R touch each other externally. Find the length of their common external tangent.

119. Two circles of radii r and R touch each other externally. A traight line l intersects the circles at points A, B, C, and D so that $AB = BC = CD$. Find AD.

120. Two circles whose radii are in the ratio 1:3 are tangent to each other externally, the length of their common external tangent being $6\sqrt{3}$ cm. Find the perimeter of the figure formed by the external tangents and external arcs of the circles.

121. Drawn from an external point to a circle are a secant whose length is 48 cm and a tangent whose length is $\frac{2}{3}$ of the internal segment of the secant. Find the radius of the circle if it is known that the secant is 24 cm distant from its centre.

122. A common external tangent to two externally touching circles forms an angle α with the line of centres. Find the ratio of the radii.

123. Drawn from the point A situated outside a circle with centre O are two secants ABC and AMK, B and M being the nearest to A points of the circle lying on the secants. Find BC if it is known that $AC = a$, $\angle CAO = \alpha$, $\angle COK = \beta$, and the secant AMK passes through the centre of the circle.

124. Two circles intersect at points A and B. Drawn through the point A are line segments AC and AD each of which, being a chord of one circle, touches the other circle. Prove that $AC^2 \cdot BD = AD^2 \cdot BC$.

125. In a circle of radius R, AB and CD are mutually perpendicular intersecting chords. Prove that $AC^2 + BD^2 = 4R^2$.

126. Prove that the sum of the squares of the distances from a point M taken on a diameter of a circle to the end points of any chord parallel to this diameter is constant for the given circle.

127. Two circles touch each other externally at a point C, AB being their common external tangent. Find their radii if $AC = 8$ cm and $BC = 6$ cm.

128. Two circles of radii R and $\frac{R}{2}$ touch each other externally. From the centre of the smaller circle, a line segment of length $2R$ is drawn at an angle of 30° to the line of centres. Find the lengths of the parts of the line segment lying outside the circles.

129. Two circles of radii a and b touch each other internally ($a < b$), the centre of the larger circle lying outside the smaller circle. A chord AB of the larger circle touches the smaller circle and forms an angle α with the common tangent to the circles. Find AB.

II. Inscribed and Circumscribed Triangles

130. Taken on the sides AB and AC of a regular triangle ABC are points M and K such that $AM:MB = 2:1$ and $AK:KC = 1:2$. Prove that the line segment KM is equal to the radius of the circle circumscribed about the triangle ABC.

131. A circle is circumscribed about a triangle ABC ($AB = BC$). When extended, the bisectors of the angles A and C intersect the circle at points K and P, and each other at a point E. Prove that the quadrilateral $BKEP$ is a rhombus.

132. In a triangle ABC, AD and CE are bisectors of the angles. The circle circumscribed about a triangle BDE passes through the centre of the circle inscribed in the triangle ABC. Prove that $\angle ABC = 60°$.

133. Prove that the centre of the circle inscribed in a triangle lies inside the triangle formed by the midlines of the given triangle.

134. A straight line l touches the circle circumscribed about a triangle ABC at a point C. Prove that the square of the altitude CH of the triangle ABC is equal to the product of the distances of the points A and B from the line l.

135. Find the angles of a triangle if it is known that the centres of its inscribed and circumscribed circles are symmetric about one of the sides of the triangle.

136. The base of an isosceles triangle is equal to $2a$, and the altitude to h. A tangent is drawn to the circle inscribed in the triangle. The tangent is parallel to the base. Find the length of the segment of this tangent enclosed between the lateral sides of the triangle.

137. In a right triangle, the point of tangency of the inscribed circle divides the hypotenuse into two segments, 24 cm and 36 cm long. Find the legs.

138. In a right triangle, one leg is equal to 48 cm, and the projection of the other leg on the hypotenuse equals 3.92 cm. Find the circumference of the inscribed circle.

139. In a right triangle with legs of 18 cm and 24 cm find the distance between the centres of the inscribed and circumscribed circles.

140. In an isosceles triangle, the altitude drawn to the base is two-thirds of the radius of the circumscribed circle. Find the base angle.

141. Find the radius of the circle circumscribed about the triangle with sides a and b and an angle γ between them.

142. In an isosceles triangle, the base is equal to b, the base angle being equal to α. A tangent is drawn to the circle inscribed in the triangle. The tangent is parallel to the base. Find the length of the segment of this tangent enclosed between the lateral sides of the triangle.

143. In an isosceles triangle, the ratio of the radii of the inscribed and circumscribed circles is equal to k. Find the angles of the triangle.

144. Prove that the following inequality holds true for any right triangle: $0.4 < \frac{r}{h} < 0.5$, where r is the radius of the inscribed circle, and h the altitude drawn to the hypotenuse.

145. Prove that the circle circumscribed about a triangle is equal to the circle passing through two of its vertices and the orthocentre.

146. A regular triangle ABC is inscribed in a circle. On the arc BC, an arbitrary point M is taken and chords AM, BM, and CM are drawn. Prove that $AM = BM + CM$.

147. Prove that the sum of the squares of the distances from an arbitrary point of a circle to the vertices of the regular triangle inscribed in this circle is a constant independent of the position of the point on the circle.

148. An isosceles triangle ABC $(AB = BC)$ is inscribed in a circle. On the arc AB, an arbitrary point K is taken and joined by chords to the vertices of the triangle. Prove that $AK \cdot KC = AB^2 - KB^2$.

149. In an acute triangle, having sides a, b, and c, perpendiculars are dropped from the centre of the circumscribed circle to the sides. The lengths of these perpendiculars are equal to m, n, and p, respectively. Prove that $\frac{m}{a} + \frac{n}{b} + \frac{p}{c} = \frac{mnp}{abc}$.

150. Prove that the feet of the perpendiculars dropped to the sides of a triangle or to their extensions from an arbitrary point of the circle circumscribed about the triangle lie in one straight line.

151. Prove that if a and b are sides of a triangle, l the bisector of the included angle, and a', b' segments into which the bisector divides the third side, then $l^2 = ab - a'b'$.

152. Prove that a radius of the circle circumscribed about a triangle drawn to one of the vertices of the triangle is perpendicular to the straight line joining the feet of the altitudes drawn from two other vertices of the triangle.

153. A circle is circumscribed about a triangle ABC. A tangent is drawn to the circle through the point B to intersect the extension of the side CA beyond the point A at the point D. Find the perimeter of the triangle ABC if $AB + AD = AC$, $CD = 3$, and $\angle BAC = 60°$.

154. A regular triangle ABC is inscribed in a circle of radius R. A chord BD intersects AC at the point E so that $AE:CE = 2:3$. Find CD.

155. In a trapezoid $ABCD$, the bisector of the angle A intersects the base BC (or its extension) at the point E. Inscribed in the triangle ABE is a circle touching the side AB at the point M and the side BE at the point P. Find the angle BAD if it is known that $AB:MP = 2$.

156. The hypotenuse of a right triangle is divided by the point of tangency of the inscribed circle into two segments whose ratio is equal to k $(k > 1)$. Find the angles of the triangle.

157. Find the angle at the base of an isosceles triangle if it is known that its orthocentre lies on the inscribed circle.

III. A Circle and a Triangle Arranged Arbitrarily

158. The line segments AD, BM, and CP are medians of a triangle ABC. The circle circumscribed about a triangle DMC passes through the centroid of the triangle ABC. Prove that $\angle ABM = \angle PCB$ and $\angle BAD = \angle PCA$.

159. A semicircle is inscribed in a right triangle so that its diameter lies on the hypotenuse, and the centre divides the hypotenuse into two segments of 15 cm and 20 cm. Find the radius of the semicircle.

160. A circle passes through the vertex A of a right triangle ABC, touches the leg BC, its centre lying on the hypotenuse AB. Find its radius if $AB = c$ and $BC = a$.

161. Constructed on the leg BC of a right triangle ABC as on the diameter is a circle intersecting the hypotenuse AB at the point D so that $AD:DB = 3:1$. Find the sides of the triangle ABC if the altitude drawn to the hypotenuse is equal to 3 cm.

162. Two sides of a triangle are equal to a and b, the angle between them being equal to 120°. Find the radius of the circle passing through two vertices (end points) of the third side and the centre of the circle inscribed in the given triangle.

163. A circle passes through the vertices A and B of a triangle ABC and touches the side BC at the point B. The side AC is divided by the circle into two parts, AM and MC, so that $AM = MC + BC$. Find BC if $AC = 4$ cm.

164. Constructed on the side AB of a triangle ABC as on the diameter is a circle intersecting the side BC at the point D. Find AC if it is known that $CD = 2$ cm and $AB = BC = 6$ cm.

165. Constructed on the side AB of a triangle ABC as on the diameter is a circle intersecting AC at the point D and BC at the point E. Find AC and BC if it is known that $AB = 3$ cm, $AD:DC = 1:1$, and $BE:EC = 7:2$.

166. The line segment BD is the altitude of a triangle ABC, and DE is a median of a triangle BCD. Inscribed in a triangle BDE is a circle touching the side BE at the point K and the side DE at the point M. Find the angles of the triangle ABC if $AB = BC = 8$ cm and $KM = 2$ cm.

167. Drawn in a triangle ABC are the altitude AD and a circle of radius AD with centre at the point A. Find the arc length of this circle lying inside the triangle if $BC = a$, $\angle B = \beta$, and $\angle C = \gamma$.

168. Prove that the radius of the circle touching the hypotenuse and the extensions of the legs of a right triangle is equal to the sum of the lengths of the hypotenuse and the radius of the circle inscribed in the triangle.

169. The angle bisectors AD and CK of a triangle ABC intersect at the point O, and $KD = 1$ cm. Find the angles and two other sides of a triangle KDO if it is known that the point B lies on the circle circumscribed about the triangle KDO.

170. A circle touches the sides AC and BC of a triangle ABC and has its centre on AB. Find the radius of the circle if $AC = 48$ cm, $BC = 140$ cm, and $AB = 148$ cm.

171. In a triangle ABC, D is the midpoint of AC, E is the midpoint of BC. The circle circumscribed about a triangle CDE passes through the centroid of the triangle ABC. Find the length of the median CK if $AB = c$.

172. Find the relationship between the sides a, b, and c of a triangle ABC if it is known that the vertex C, the centroid M, and the midpoints of the sides AC and BC lie in one and the same circle.

173. Inscribed in an isosceles triangle ABC with the angle B equal to 120° is a semicircle of radius $(3\sqrt{3} + \sqrt{21})$ cm with centre on AC. Drawn to the semicircle is a tangent intersecting the lateral sides AB and BC at points D and E, respectively. Find BD and BE if $DE = 2\sqrt{7}$ cm.

174. In a triangle ABC, three sides are known, i.e. $AB = BC = 39$ cm, $AC = 30$ cm, and the altitudes AD and BE are drawn. Find the radius of the circle passing through the points D and E and touching the side BC.

175. In a triangle ABC, the altitude CD and AE are drawn. A circle is circumscribed about a triangle BDE. Find the arc length of this circle lying inside the triangle ABC if $AC = b$ and $\angle ABC = \beta$.

IV. A Circle and a Quadrilateral

176. Prove that if there are an inscribed and a circumscribed circle for a trapezoid, then the altitude of the trapezoid is the geometric mean between its bases.

177. The bases of an isosceles trapezoid are equal to 21 cm and 9 cm, and the altitude to 8 cm. Find the radius of the circumscribed circle.

178. The bases of an isosceles trapezoid are a and b, and the acute angle is α. Find the radius of the circumscribed circle.

179. Two vertices of a square lie on a circle of radius R, and two others on a tangent to this circle. Find the side of the square.

180. The acute angle A of a rhombus $ABCD$ is equal to α. Find the ratio of the radius of the circle inscribed in the rhombus to that of the circle inscribed in the triangle ABC.

181. An isosceles trapezoid is circumscribed about a circle. Find its angles if it is known that the ratio of the lateral side of the trapezoid to its smaller base is equal to k.

182. Circumscribed about a circle is a trapezoid with acute angles α and β. Find the ratio of the perimeter of the trapezoid to the circumference of the circle.

183. (a) Prove Ptolemy's theorem: a necessary and sufficient condition that a convex quadrilateral be inscribable in a circle is that the sum of the products of the two pairs of opposite sides be equal to the product of the diagonals; in other words, if the opposite sides of a quadrilateral inscribed in a circle are equal to a and b, c and m, and the diagonals to d_1 and d_2, then $ab + cm = d_1d_2$; (b) taking advantage of Ptolemy's theorem, prove the statement of Problem 146.

184. Prove that the sum of the products of the altitudes of an acute triangle and their segments from the orthocentre to a vertex is equal to half the sum of the squares of the sides.

185. Constructed on the hypotenuse of a right triangle as on a side is a square (outside the triangle). The centre of the square is joined to the vertex of the right angle of the triangle. Into what segments is the hypotenuse divided if the legs are equal to 21 cm and 28 cm?

186. A circle touches two adjacent sides of a square and divides each of the two other of its sides into two segments of 2 cm and 23 cm. Find the radius of the circle.

187. Inscribed in a rhombus $ABCD$ 4 cm on a side and the angle BAD equal to $60°$ is a circle. Drawn to this circle is a tangent intersecting AB at the point M and AD at the point P. Find MB and PD if $MP = 2$ cm.

188. The ratio of the radius of the circle circumscribed about a trapezoid to the radius of the inscribed circle is equal to k. Find the acute angle of the trapezoid.

189. Inscribed in a circle is a quadrilateral $ABCD$ whose diagonals are mutually perpendicular and intersect at the point E. The straight line passing through the point E perpendicular to AB intersects CD at the point M. Find EM if $AD = 8$ cm, $AB = 4$ cm, and $\angle CDB = \alpha$.

190. Inscribed in a circle is a quadrilateral $ABCD$ whose diagonals are mutually perpendicular and intersect at the point E. The straight line passing through the point E and the midpoint of the side CD intersects AB at the point H. Find HB if $ED = 6$ cm, $BE = 5$ cm, and $\angle ADB = \alpha$.

191. In a convex quadrilateral $ABCD$, the side AB is equal to $\frac{25}{64}$, the side BC to $12\frac{25}{64}$, the side CD to $6\frac{1}{4}$. It is known that the angle DAB is acute, the angle ADC is obtuse, $\sin\angle DAB = \frac{3}{5}$, and $\cos\angle ABC = -\frac{63}{65}$. A circle with centre at O touches the sides BC, CD, and AD. Find the length of the line segment OC.

V. Miscellaneous Problems

192. Drawn from the point C to a circle are two tangents CA and CB forming an angle of $60°$. A circle is inscribed in the curvilinear triangle formed by these tangents and the minor arc AB. Prove that the length of this arc is equal to the circumference of the inscribed circle.

193. A rectangle, having sides 36 cm and 48 cm, is separated into two triangles by the diagonal. A circle is inscribed in each of these triangles. Find the distance between their centres.

194. Two circles of radii 16 cm and 9 cm touch each other externally. Compute the radius of the circle inscribed in the curvilinear triangle enclosed between the circles and their common external tangent.

195. A chord, 6 cm long, separates a circle into two segments. A square of 2 cm on a side is inscribed in the smaller segment. Find the radius of the circle.

196. Two circles of radius R are arranged so that the distance between their centres is equal to R. A square is inscribed in the intersection of the circles. Find the side of the square.

197. A circle is inscribed in a sector of a circle having an angle of 2α. Find the ratio of the radii of the inscribed circle and the sector.

198. Inscribed in the sector AOB of a circle of radius R with a central angle α is a regular triangle one of whose vertices lies at the middle of the arc AB, and two others on the radii OA and OB. Find the side of the triangle.

199. An arc of a circle of radius R subtends a central angle 2α $\left(\alpha < \frac{\pi}{2}\right)$. The chord of this arc divides the circle into two segments. Inscribed in the smaller segment is a square. Find the side of the square.

200. An arc of a circle of radius R subtends a central angle 2α $\left(\alpha < \frac{\pi}{2}\right)$. The chord of this arc divides the circle into two segments. Inscribed in the smaller segment is a regular triangle so that one of its vertices coincides with the midpoint of the arc, and two other vertices lie on the chord of the segment. Find the side of the triangle.

201. An arc of a circle of radius R subtends a central angle 2α $\left(\alpha < \frac{\pi}{2}\right)$. The chord of this arc divides the circle into two segments. Inscribed in the larger segment is a regular triangle so that one of its vertices coincides with the midpoint of the chord, and two others lie on the arc. Find the side of the triangle.

202. A circle of radius a is inscribed in an isosceles triangle. A circle of radius b touches the lateral sides of the triangle and the inscribed circle. Find the base of the triangle.

203. A point B is taken on the line segment AC whose length is equal to 12 cm so that $AB = 4$ cm. Circles are constructed on AC and AB as on diameters. Find the radius of the circle touching the two constructed circles and the segment AC.

204. The base of an isosceles triangle is b, and the base angle is α. A circle is inscribed in the triangle. A second circle touches the first circle and the lateral sides of the triangle. Find the radius of the second circle.

205. Two radii (OA and OB) are drawn in a circle of radius R with centre at O so that $\angle AOB = \alpha$ $\left(\frac{\pi}{2} < \alpha < \pi\right)$. Find the radius of the circle touching the arc AB of the sector OAB, the chord AB, and the bisector of the angle AOB.

206. Two equal circles of radius a are arranged so that the distance between their centres is equal to a. The intersection of the circles is divided by the line of centres into two curvilinear triangles, a circle being inscribed in one of them. Find the length of the line segment joining the points of tangency of the inscribed circle and the two given circles.

207. A tangent AK is drawn from the point A to the circle of radius of 2 cm with centre at point O. The line segment OA intersects the circle at the point M and forms an angle of 60° with the tangent. Find the radius of the circle inscribed in the curvilinear triangle MKA.

208. Drawn from the point A situated at a distance a from the centre O of a circle of radius r $(a > r)$ is a ray forming an angle of 60° with the ray AO and intersecting the circle at two points K and P (K lying between A and P). Find the radius of the circle inscribed in the curvilinear triangle MKA, where M is the point of intersection of the circle and the line segment AO.

209. The base of an isosceles triangle is b, and the base angle is α. A circle is inscribed in the triangle. Another circle touches the first circle, the base, and a lateral side of the triangle. Find the radius of the second circle.

210. A circle is circumscribed about an isosceles triangle with the base b and the base angle α. Another circle touches the first circle and the lateral sides of the triangle. Find the radius of the second circle.

211. Inscribed in a segment of a circle of radius R with the central angle α $(\alpha < \pi)$ are two equal circles touching each other. Find their radii.

212. Points D, K, and M lie on the respective sides AB, BC, and AC of a triangle ABC. Prove that the circles circumscribed about the triangles ADM, BDK, and CKM intersect at a common point.

213. Two tangents AC and BC are drawn from the point C to the circle of radius of 12 cm with centre at point O. Inscribed in a triangle ABC is a circle with centre at O_1 touching the sides AC and BC at points K and H. Find the angle AOB if the distance from the point O_1 to the straight line KH is equal to 3 cm.

214. Drawn from the centre O of a circle of radius R are two radii OA and OB so that $\angle AOB = \alpha$ $(\alpha < \pi)$. Inscribed in the smaller segment of the circle cut off by the chord AB is a regular triangle one of the sides of which is perpendicular to the chord AB. Find the side of the triangle.

215. A diameter AB and a chord AC are drawn in a circle of radius r. A circle is inscribed in the curvilinear triangle thus formed. Find its radius if $\angle CAB = \alpha$.

216. A radius OM and a chord KP intersect at the point A in a circle with centre O, and $\angle MAK = \alpha$ $\left(\alpha < \frac{\pi}{2}\right)$. A circle is inscribed in the curvilinear triangle thus formed. Find its radius if $OM = r$ and $OA = a$.

217. Drawn from the point A of a circle of radius r are a diameter AD and two chords AB and AC. Find the radius of the circle touching the arc BC and the chords AB and AC if $AB = b$, $\angle BAC = \alpha$, and $AB > AC$.

218. Given on one side of an angle α are two points, the distances of which from the other side of the angle are equal to b and c $(b < c)$. Find the radius of the circle passing through these two points and touching the other side of the angle.

219. The angle AOB is equal to α. A circle touches the side AO at a point C and intersects the side OB at points D and E. Find DE and the radius of the circle if it is known that $OC = a$ and $OD = b$ $(b > a)$.

SEC. 4. AREAS OF PLANE FIGURES

Example 1. The point H is the orthocentre of the triangle ABC. A point K is taken on the straight line CH such that ABK is a right triangle. Prove that the area of the triangle ABK is the geometric mean between the areas of the triangles ABC and ABH (Fig. 55).

Solution. We introduce the following notation: $S_{ABK} = S$, $S_{ABC} = S_1$, and $S_{ABH} = S_2$. Then $S = \frac{1}{2} AB\cdot KD$, $S_1 = \frac{1}{2} AB\cdot CD$, and $S_2 = \frac{1}{2} AB\cdot HD$. We have to prove that

$$S = \sqrt{S_1 S_2}\,, \tag{1}$$

i.e. that $\frac{1}{2} AB\cdot KD = \sqrt{\frac{1}{2} AB\cdot CD\cdot \frac{1}{2} AB\cdot HD}$ or that

$$KD^2 = CD\cdot HD. \tag{2}$$

But ABK is a right triangle, and, therefore, $KD^2 = BD\cdot AD$ (Theorem 6a). Thus, Equality (2) will be ascertained if we prove that $BD\cdot AD = CD\cdot DH$, or that $\frac{BD}{CD} = \frac{DH}{AD}$.

The last equality obviously follows from the similarity of the right triangles BCD and HDA (in these triangles the angles BCD and HAD are equal as angles with mutually perpendicular sides since AE is the altitude of the triangle). Hence, Equality (2) as well as Equality (1) have been proved.

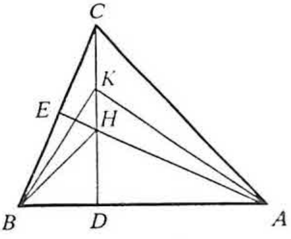


Fig. 55

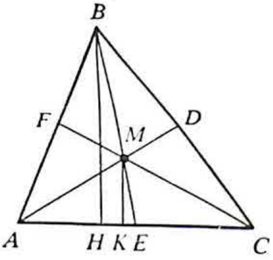


Fig. 56

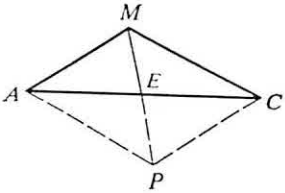


Fig. 57

Example 2. Given the medians m_a, m_b, and m_c of a triangle, compute its area.

Solution. First of all note that $S_{AMC} = \frac{1}{3} S_{ABC}$ (Fig. 56). Indeed, these triangles have a common base AC, and, hence, their areas are to each other as the altitudes MK and BH (Theorem 18). But from the similarity of the triangles MKE and BHE we conclude that $\frac{MK}{BH} = \frac{ME}{BE}$ and $ME:BE = 1:3$ (Theorem 3b). Thus, the sought area S is equal to $3S_{AMC}$.

Consider the triangle AMC (Fig. 57). Two of its sides, $AM = \frac{2}{3} m_a$, $MC = \frac{2}{3} m_c$, and the median $ME = \frac{1}{3} m_b$ are known (we use Theorem 3 once again). We lay off EP equal to ME and join P to A and C to get a parallelogram $MCPA$. We obtain: $S_{AMC} = S_{MCP} = \frac{1}{2} S_{AMCP}$. Three sides, i.e. $\frac{2}{3} m_a$, $\frac{2}{3} m_b$, and $\frac{2}{3} m_c$, are known in the triangle MCP. Hence, the area of the triangle MCP can be found by Hero's formula (Theorem 19e).

Thus,

$$S = 3S_{AMC} = 3S_{MCP}$$

$$= 3\sqrt{\frac{1}{3}(m_a + m_b + m_c)\,\frac{1}{3}(m_a + m_b - m_c)\,\frac{1}{3}(m_a + m_c - m_b)}$$

$$\times \sqrt{\frac{1}{3}(m_b + m_c - m_a)} = \frac{1}{3}\sqrt{(m_a + m_b + m_c)(m_a + m_b - m_c)}$$

$$\times \sqrt{(m_a + m_c - m_b)(m_b + m_c - m_a)}.$$

Example 3. Find the area of a triangle with angles α, β, and γ knowing that the distances from an arbitrary point M taken inside the triangle to its sides are equal to m, n, and k (Fig. 58).

Solution. The area S of the triangle ABC can be found by the formula $S = \frac{1}{2} AC \cdot BC \cdot \sin\gamma$, for which purpose we have to find AC and BC. Let us set $BC = x$. Then, by the law of sines (Theorem 8), we have: $\frac{AC}{\sin\beta} = \frac{BC}{\sin\alpha} = \frac{AB}{\sin\gamma}$, whence we find that $AC = \frac{x \sin\beta}{\sin\alpha}$ and $AB = \frac{x \sin\gamma}{\sin\alpha}$.

Thus, the problem is reduced to finding x. To set up an equation, we are going to apply the method of areas (see Sec. 1), taking the area S of the triangle ABC as a reference element.

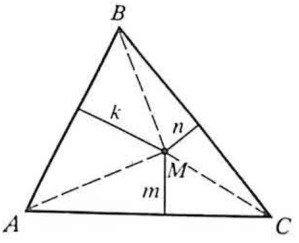


Fig. 58

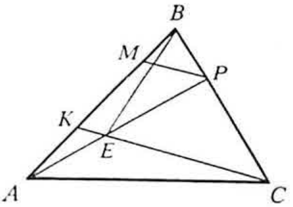


Fig. 59

On the one hand, we have: $S=\frac{1}{2}AC\cdot BC\cdot\sin\gamma=\frac{1}{2}\cdot\frac{x\sin\beta}{\sin\alpha}\times x\sin\gamma=\frac{x^2\sin\beta\sin\gamma}{2\sin\alpha}$. On the other hand, $S=S_{AMB}+S_{BMC}+S_{AMC}=\frac{1}{2}AB\cdot k+\frac{1}{2}BC\cdot n+\frac{1}{2}AC\cdot m=\frac{1}{2}\cdot\frac{x\sin\gamma}{\sin\alpha}\cdot k+\frac{1}{2}xn+\frac{1}{2}\cdot\frac{\sin\beta}{\sin\alpha}\cdot m=\frac{x(k\sin\gamma+n\sin\alpha+m\sin\beta)}{2\sin\alpha}$.

Hence, $\frac{x^2\sin\beta\sin\gamma}{2\sin\alpha}=\frac{x(k\sin\gamma+n\sin\alpha+m\sin\beta)}{2\sin\alpha}$, whence we get: $x=\frac{k\sin\gamma+n\sin\alpha+m\sin\beta}{\sin\beta\sin\gamma}$.

Substituting this value of x into the first of the above formulas for the area of the triangle ABC, we obtain: $S=\frac{x^2\sin\beta\sin\gamma}{2\sin\alpha}=\frac{(k\sin\gamma+n\sin\alpha+m\sin\beta)^2}{2\sin\alpha\sin\beta\sin\gamma}$.

Example 4. Taken on the sides AB and BC of the triangle ABC are points K and P such that $AK:BK=1:2$ and $CP:BP=2:1$. The straight lines AP and CK intersect at the point E. Find the area of the triangle ABC if it is known that the area of the triangle BEC is equal to 4 cm² (Fig. 59).

Solution. We set $AK=x$, $BK=2x$, $BP=y$, $CP=2y$ and draw $PM\parallel KC$. By the Thales theorem, $\frac{BM}{MK}=\frac{BP}{PC}=\frac{1}{2}$. Hence, $BM=\frac{2x}{3}$ and $KM=\frac{4x}{3}$.

Further, the triangles AKE and AMP are similar, therefore, $\frac{KE}{MP}=\frac{AK}{AM}$, i.e. $\frac{KE}{MP}=\frac{x}{x+\frac{4x}{3}}=\frac{3}{7}$, and, hence, $KE=\frac{3}{7}MP$.

On the other hand, $\frac{MP}{KC}=\frac{BP}{BC}=\frac{1}{3}$, i.e. $MP=\frac{1}{3}KC$.

Thus, we get: $KE=\frac{1}{7}KC$, and, therefore, $EC=\frac{6}{7}KC$.

Consider the triangles BEC and BKC. They have a common altitude (the one drawn from the vertex B), and, hence, their areas are

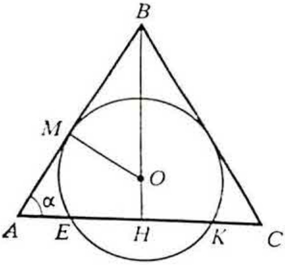


Fig. 60

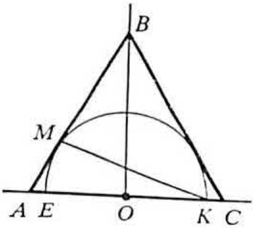


Fig. 61

to each other as the bases KC and EC (Theorem 18), that is, $\frac{S_{BKC}}{S_{BEC}}=\frac{KC}{EC}=\frac{7}{6}$. But $S_{BEC}=4$ cm², and, consequently, $S_{BKC}=\frac{7}{6}\cdot 4=\frac{14}{3}$ cm².

Finally, consider the triangles BKC and ABC. They have a common altitude (the one drawn from the vertex C), and, hence, the ratio of their areas is equal to the ratio of their bases: $\frac{S_{ABC}}{S_{BKC}}=\frac{AB}{BK}=\frac{3x}{2x}=\frac{3}{2}$.

Thus, we get: $S_{ABC}=\frac{3}{2}S_{BKC}=\frac{3}{2}\times\frac{14}{3}=7$ cm².

Example 5. The angle A of the triangle ABC $(AB=BC)$ is equal to $\arctan\frac{8}{15}$ (Fig. 60). A circle of radius of 1 cm touches the sides AB and BC and intersects the base AC at points E and K (E lying between A and K), M is the point of contact of the circle and the straight line BA, and $AM=\frac{15}{8}$ cm. Compute the area of the triangle AMK.

Solution. First of all, we have to carry out computations which will enable us to find out where the centre of the circle lies (for the time being it is clear only that this centre lies on the altitude BH of the isosceles triangle ABC, since BA and BC are tangents to the circle, and, therefore, the centre of the circle lies on the bisector of the angle between the lines) (Theorem 12b).

Let us denote the angle BAC by α. We draw a radius OM in the point of tangency, then the angle BOM is also equal to α. By the hypothesis, $\tan\alpha=\frac{8}{15}$. Taking advantage of the formula $1+\tan^2\alpha=\frac{1}{\cos^2\alpha}$, we find that $\cos\alpha=\frac{15}{17}$, then $\sin\alpha=\tan\alpha\cos\alpha=\frac{8}{17}$.

We find from the triangle BOM that $BO=\frac{OM}{\cos\alpha}=\frac{1}{\frac{15}{17}}=\frac{17}{15}$ and $BM=OM\tan\alpha=\frac{8}{15}$. Further, $AB=AM+BM=\frac{15}{8}+\frac{8}{15}=\frac{289}{120}$ and $BH=AB\sin\alpha=\frac{289}{120}\cdot\frac{8}{17}=\frac{17}{15}$.

This means that $BH=BO$, and, therefore, the points O and H coincide, and for the further solution of the problem we have to make a new (correct) drawing (Fig. 61).

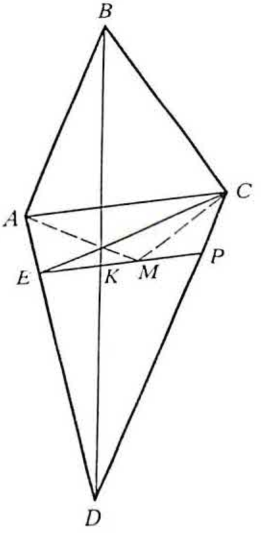


Fig. 62

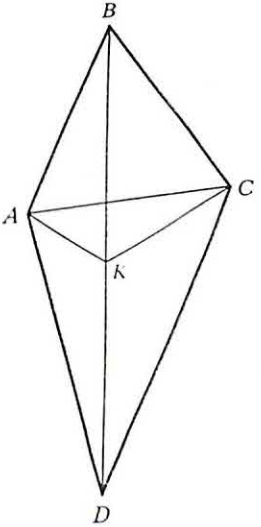


Fig. 63

We shall determine the area of the triangle AMK with the aid of the formula $S=\frac{1}{2}AM\cdot AK\cdot\sin\alpha$. It is known that $AM=\frac{15}{8}$ cm and $\sin\alpha=\frac{8}{15}$. Thus, the problem has been reduced to finding the line segment AK.

Let us take advantage of the fact that $AM^2=AE\cdot AK$ (Theorem 16c). We set $AE=x$, then $AK=2+x$, and we get the equation: $\frac{225}{64}=x(2+x)$, whence $x=\frac{9}{8}$.

Then $AK=\frac{9}{8}+2=\frac{25}{8}$ cm, and, consequently, $S_{AMK}=\frac{1}{2}AM\cdot AK\cdot\sin\alpha=\frac{1}{2}\cdot\frac{15}{8}\cdot\frac{25}{8}\cdot\frac{8}{17}=\frac{375}{272}$ cm^2.

Example 6. Drawn through the midpoint of the diagonal BD in the quadrilateral $ABCD$ is a straight line parallel to the diagonal AC. This line intersects the side AD at the point E. Prove that the line CE divides the quadrilateral $ABCD$ into two equivalent parts (Fig. 62).

Solution. We have to prove that the area of the quadrilateral $ABCE$ is equal to half the area of the quadrilateral $ABCD$. This will just mean that the areas of two figures $ABCE$ and CED are equal, that is, that the figures are equivalent.

Note that the quadrilateral $ABCE$ is equivalent to the quadrilateral $ABCM$, where M is any point on the line EP: indeed, the triangles ACE and ACM have a common base and equal altitudes since the points E and M lie in a straight line parallel to the base AC. This fact prompts the idea of replacing the quadrilateral $ABCE$ by an equivalent quadrilateral $ABCK$, where K is a specially chosen point on EP. Let us choose the middle of the diagonal BD as K (Fig. 63). We have: $S_{ABCK} = \frac{1}{2} AC \cdot BK \cdot \sin\alpha$, where α is an angle between the diagonals (Theorem 20b). By the hypothesis, $BK = \frac{1}{2} BD$.

Thus, $S_{ABCK} = \frac{1}{2} AC \cdot \frac{1}{2} BD \cdot \sin\alpha = \frac{1}{2}\left(\frac{1}{2} AC \cdot BD \cdot \sin\alpha\right) = \frac{1}{2} S_{ABCD}$, which was required to be proved.

Example 7. The area of a convex quadrilateral $ABCD$ is equal to 2 cm^2. Its sides are extended: the side AB beyond the point B so that $BL = \frac{1}{2} AB$, the side BC beyond the point C so that $CP = \frac{1}{2} BC$, the side CD beyond the point D so that $DE = \frac{1}{2} CD$, and the side DA beyond the point A so that $AM = \frac{1}{2} AD$. Find the area of the quadrilateral $MLPE$ (Fig. 64).

Solution. Introduce the notation: $AB = a$, $BC = b$, $CD = c$, and $DA = m$. Consider the triangle AML. In it, $AM = \frac{1}{2} m$ and $AL = \frac{3}{2} a$. Then its area S_1 is equal to $\frac{1}{2} \cdot \frac{1}{2} m \cdot \frac{3}{2} a \cdot \sin\alpha$, where $\alpha = \angle MAL$. Compare this expression with the area of the triangle ABD: $S_{ABD} = \frac{1}{2} AB \cdot AD \cdot \sin(180^\circ - \alpha) = \frac{1}{2} am \sin\alpha$. Note that $S_1 = \frac{3}{4} S_{ABD}$.

Analogously, the area S_3 of the triangle CPE is related to the area of the triangle BCD by the formula $S_3 = \frac{3}{4} S_{BCD}$. Hence, $S_1 + S_3 = \frac{3}{4}(S_{ABD} + S_{BCD}) = \frac{3}{4} S_{ABCD} = \frac{3}{4} \cdot 2 = 1.5$ cm^2. In the same way, if we set $S_2 = S_{BLP}$ and $S_4 = S_{MDE}$, we obtain: $S_2 + S_4 = 1.5$ cm^2.

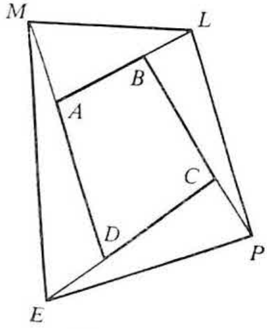


Fig. 64

Hence, $S_{MLPE} = S_{ABCD} + S_1 + S_2 + S_3 + S_4 = 2 + 1.5 + 1.5 = 5$ cm^2.

Example 8. A circle with centre O is circumscribed about the triangle ABC with an obtuse angle A. The radius AO forms an angle of 30° with the altitude AH. The extension of the angle bisector AF intersects the circle at the point L, and the radius AO intersects BC at the point E (Fig. 65). Compute the area of the quadrilateral

FEOL if it is known that $AL = 4\sqrt{2}$ cm and $AH = \sqrt{2\sqrt{3}}$ cm.

Solution. In Example 6, we determined the area of a quadrilateral using Theorem 20b, and in Example 7, as the sum of the areas of component parts. In the present case, it is advisable to consider the quadrilateral in question as the difference of the triangles AOL and AFE. Hence, $S_{FEOL} = S_{AOL} - S_{AFE}$. Therefore, the further solution of the problem will be mainly reduced to computing various elements (sides, angles) of the triangles AOL and AFE.

We are going to prove that OL is parallel to AH. To this end, note that $\angle CAL = \angle LAB$ (by the hypothesis), and, therefore, $\smile CL = \smile BL$. But then the chords CL and BL are also equal, that is, CBL is an isosceles triangle (Fig. 66). But the centre O of the circle circumscribed about the isosceles triangle CBL lies on its altitude KL. It is obvious that $KL \parallel AH$, and, therefore, $OL \parallel AH$. But then $\angle HAF = \angle ALO = \angle LAO = \frac{1}{2}\angle HAO = \frac{1}{2}\cdot 30^\circ = 15^\circ$.

And so, let us sum up, as a preliminary: we know that AOL is an isosceles triangle with angles 15°, 15°, and 150° and side AL equal to $4\sqrt{2}$ cm. This is sufficient to compute its area.

By the law of cosines (Theorem 7), we have: $AL^2 = AO^2 + OL^2 - 2AO\cdot OL\cdot\cos 150^\circ$, whence, setting $AO = OL = R$, we get: $(4\sqrt{2})^2 = R^2 + R^2 + 2R^2\frac{\sqrt{3}}{2}$, $R^2 = \frac{32}{2+\sqrt{3}} = 32\,(2-\sqrt{3})$.

Further, we have: $S_{AOL} = \frac{1}{2}AO\cdot OL\cdot\sin 150^\circ = \frac{1}{2}R^2\cdot\frac{1}{2} = \frac{1}{4}\times 32\,(2-\sqrt{3}) = 8\,(2-\sqrt{3})$ cm².

Now, compute the area of the triangle AFE. We have: $HE = AH\tan 30^\circ = \sqrt{2\sqrt{3}}\cdot\frac{\sqrt{3}}{3}$, $HF = AH\tan 15^\circ = \sqrt{2\sqrt{3}}\,(2-\sqrt{3})$, $FE = HE - HF = \sqrt{2\sqrt{3}}\left(\frac{\sqrt{3}}{3} - 2 + \sqrt{3}\right) = \sqrt{2\sqrt{3}}\,\frac{2\,(2-\sqrt{3})}{\sqrt{3}}$, and $S_{AFE} = \frac{1}{2}FE\cdot AH = \frac{1}{2}\sqrt{2\sqrt{3}}\,\frac{2\,(2-\sqrt{3})}{\sqrt{3}}\sqrt{2\sqrt{3}} = 2\,(2-\sqrt{3})$ cm².

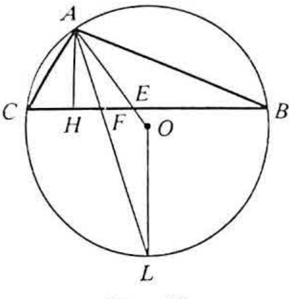


Fig. 65

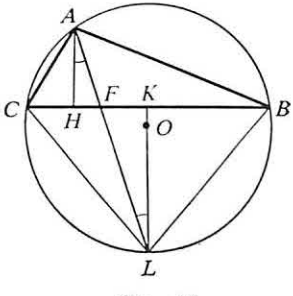


Fig. 66

Hence, $S_{FEOL} = S_{AOL} - S_{AFE} = 8(2-\sqrt{3}) - 2(2-\sqrt{3}) = 6(2-\sqrt{3})$ cm^2.

Example 9. In the pentagon $ABCDE$, $AB=\sqrt{2}$, $BC=CD$, $\angle ABE = 45^\circ$, and $\angle DBE = 30^\circ$ (Fig. 67). Compute the area of the pentagon if a circle of radius of 1 cm can be circumscribed about this pentagon.

Solution. We are going to compute the area of the given pentagon as the sum of the areas of the triangles ABE, BDE, and BCD. By the law of sines, applied to the triangle ABE, we find that $\frac{AE}{\sin 45^\circ} = 2R$, that is, $AE = \sqrt{2}$. Hence, ABE is a right isosceles triangle in which $AB = AE = \sqrt{2}$, and, therefore, $BE = 2$ and $S_{ABE} = \frac{1}{2} AB\cdot AE = 1$.

Since $BE = 2$, BE is a diameter of the circle. Hence, BDE is a right triangle, wherefrom we find that $DE = 1$, $BD = \sqrt{3}$, and $S_{BDE} = \frac{1}{2} BD\cdot DE = \frac{\sqrt{3}}{2}$.

Finally, consider the triangle BCD, where $BD = \sqrt{3}$. By the law of sines, $\frac{BD}{\sin \angle BCD} = 2R$, that is, $\sin \angle BCD = \frac{\sqrt{3}}{2}$. Hence we obtain: $\angle BCD = 120^\circ$ and $\angle CBD = \angle CDB = 30^\circ$. Since $\frac{BC}{\sin 30^\circ} = 2R$, we get: $BC = CD = 1$ and $S_{BCD} = \frac{1}{2} BC\cdot CD\cdot \sin\angle BCD = \frac{1}{2}\cdot 1\cdot 1\cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$.

Thus, $S_5 = 1 + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4} = \frac{4+3\sqrt{3}}{4}$.

Example 10. The centres of four circles are situated at the vertices of a square with side a, each radius being equal to a. Compute the area of the intersection of the circles (Fig. 68).

Solution. For symmetry reasons, it follows that the quadrilateral $EKMP$ is a square. Hence, the desired figure is a combination of a

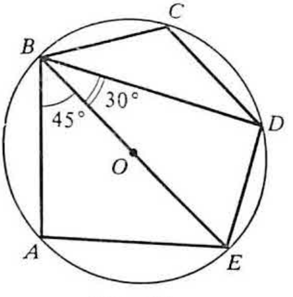


Fig. 67

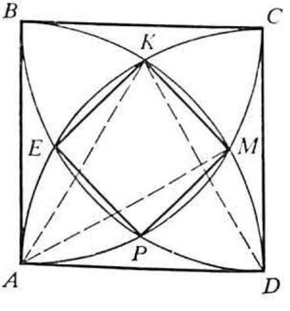


Fig. 68

square and four equal segments. To compute the area of a segment, we have, first of all, to find the corresponding central angle. Since the triangle AKD is equilateral, $\angle KAD = 60^\circ$, and, hence, $\angle BAK = 30^\circ$. Similarly, we get: $\angle MAD = 30^\circ$, and, therefore, $\angle KAM = 30^\circ$. Applying Theorem 24, we find the area of the segment: $S_{\text{segm}} = \frac{1}{2}a^2\left(\frac{\pi}{6} - \frac{1}{2}\right)$.

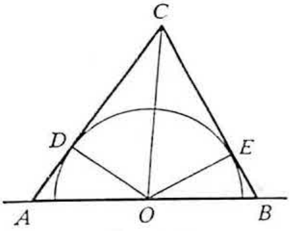


Fig. 69

To find the side of the square $EKMP$, we apply the law of cosines to the triangle AKM: $KM^2 = AK^2 + AM^2 - 2AK\cdot AM\cdot\cos 30^\circ$, i.e. $KM^2 = a^2 + a^2 - 2a^2\cdot\frac{\sqrt{3}}{2} = a^2(2 - \sqrt{3})$.

Finally, we get: $S = S_{\text{square}} + 4S_{\text{segm}} = a^2(2 - \sqrt{3}) + 2a^2\left(\frac{\pi}{6} - \frac{1}{2}\right) = a^2\left(1 + \frac{\pi}{3} - \sqrt{3}\right)$.

Example 11. A circle touches the sides AC and BC of the triangle ABC at points D and E, respectively, and has its centre on the side AB. Find the area of the sector DOE if $BC = 13$ cm, $AB = 14$ cm, and $AC = 15$ cm (Fig. 69).

Solution. To find the radius of the sector, we are going to apply the method of areas (see Sec. 1). On the one hand, the area S of the triangle ABC can be found by Hero's formula (Theorem 19e): $S = 84$ cm². On the other hand, $S = S_{AOC} + S_{BOC} = \frac{1}{2}AC\cdot DO + \frac{1}{2}BC\cdot OE = \frac{1}{2}(15 + 13)r = 14r$.

Hence, $14r = 84$ and $r = 6$ cm.

To find the area of a sector, it is necessary to know its central angle, that is, the angle DOE. From the quadrilateral $ODCE$ we conclude that $\angle DOE = \pi - \gamma$, where $\gamma = \angle ACB$. By the law of cosines, $AB^2 = AC^2 + BC^2 - 2AC\cdot BC\cdot\cos\gamma$. Hence, $14^2 = 13^2 + 15^2 - 2\cdot 13\cdot 15\cdot\cos\gamma$, whence we find that $\cos\gamma = \frac{99}{195}$, and, therefore, $\gamma = \arccos\frac{99}{195}$. Hence it follows that the central angle of the sector is equal to $\pi - \arccos\frac{99}{195}$.

By Theorem 23, we get: $S_{\text{sector}} = \frac{1}{2}r^2\left(\pi - \arccos\frac{99}{195}\right) = 18\left(\pi - \arccos\frac{99}{195}\right)$.

Example 12. A point M is taken inside the triangle ABC with sides a, b, and c so that the sides of the triangle are seen at equal angles from this point. Find $AM + BM + CM$ (Fig. 70).

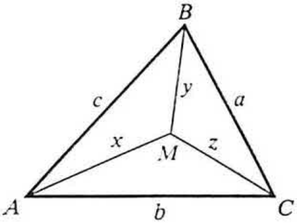


Fig. 70

Solution. As distinct from the previous problems, this problem does not deal with the computation of the area of a plane figure. Nevertheless, as we shall see, the area of a triangle will turn out to be a means for solution.

Let us set $AM = x$, $BM = y$, and $CM = z$. By the hypothesis, $\angle AMB = \angle BMC = \angle AMC = 120°$. Applying the law of cosines to each of the triangles AMB, BMC, and AMC, we get the system of equations:

$$\begin{cases} a^2 = z^2 + y^2 + yz, \\ b^2 = x^2 + z^2 + xz, \\ c^2 = x^2 + y^2 + xy. \end{cases}$$

Further, we have: $S = S_{ABC} = S_{AMC} + S_{BMC} + S_{AMB} = \frac{1}{2} xz \sin 120° + \frac{1}{2} yz \sin 120° + \frac{1}{2} xy \sin 120° = \frac{\sqrt{3}}{4}(xz + yz + xy)$. Hence, $xy + xz + yz = \frac{4S}{\sqrt{3}}$, where $S = \sqrt{p(p-a)(p-b)(p-c)}$ $\left(p = \frac{a+b+c}{2}\right)$.

It is necessary to find the value of the sum: $x + y + z$. We have: $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$. Adding the three equations of the system, we get: $x^2 + y^2 + z^2 = \frac{a^2+b^2+c^2}{2} - \frac{1}{2}(xy + xz + yz)$.

Hence, $(x+y+z)^2 = \frac{a^2+b^2+c^2}{2} + \frac{3}{2}(xy + xz + yz) = \frac{a^2+b^2+c^2}{2} + \frac{3}{2} \cdot \frac{4S}{\sqrt{3}} = \frac{a^2+b^2+c^2}{2} + 2S\sqrt{3}$, and, consequently, $x + y + z = \sqrt{\frac{a^2+b^2+c^2}{2} + 2S\sqrt{3}}$.

Example 13. In the triangle ABC, $AC:BC = 2:1$ and $\angle C = \arccos \frac{3}{4}$. Taken on the side AC is a point D such that $CD:AD = 1:3$. Find the ratio of the radius of the circle circumscribed about the triangle ABC to the radius of the circle inscribed in the triangle ABD.

Solution. We introduce an auxiliary parameter, i.e. $CD = a$. Then $AD = 3a$, $AC = 4a$, and $BC = 2a$ (Fig. 71).

In order to find the radius R of the circle circumscribed about the triangle ABC, we are going to compute the side AB by the law of cosines, and then use the law of sines. We have: $AB^2 = AC^2 + BC^2 - 2AC \cdot BC \cdot \cos C$, that is, $AB^2 = 16a^2 + 4a^2 - 2 \cdot 4a \cdot 2a \cdot \frac{3}{4}$, whence we find: $AB = 2a\sqrt{2}$. By the hypothesis, $\cos C = \frac{3}{4}$, and, hence, $\sin C = \sqrt{1-\cos^2 C} = \frac{\sqrt{7}}{4}$. By the law of sines, $\frac{AB}{\sin C} = 2R$, and, hence, $\frac{2a\sqrt{2}\cdot 4}{\sqrt{7}} = 2R$, whence we find: $R = \frac{4a\sqrt{2}}{\sqrt{7}}$.

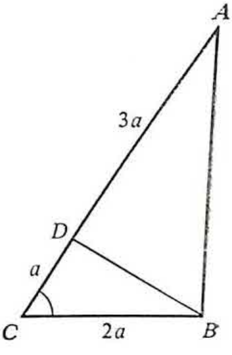


Fig. 71

The radius r of the circle inscribed in the triangle ABD is found by the formula $r = \frac{S}{p}$, where S is the area, and p the semiperimeter of the triangle ABD. It is already known that $AD = 3a$ and $AB = 2a\sqrt{2}$. The side BD is found from the triangle BCD by the law of cosines: $BD^2 = a^2 + 4a^2 - 2\cdot a\cdot 2a\cdot\frac{3}{4}$, whence $BD = a\sqrt{2}$. Hence,

$$p = \frac{3a+2a\sqrt{2}+a\sqrt{2}}{2} = \frac{3a}{2} + \frac{3a\sqrt{2}}{2}.$$

The area S of the triangle ABD is computed by Hero's formula:

$$S = \sqrt{p(p-AD)(p-AB)(p-BD)}$$

$$= \sqrt{\left(\frac{3a}{2}+\frac{3a\sqrt{2}}{2}\right)\left(\frac{3a\sqrt{2}}{2}-\frac{3a}{2}\right)\left(\frac{3a}{2}-\frac{a\sqrt{2}}{2}\right)\left(\frac{3a}{2}+\frac{a\sqrt{2}}{2}\right)}$$

$$= \sqrt{\left(\frac{9a^2}{2}-\frac{9a^2}{4}\right)\left(\frac{9a^2}{4}-\frac{a^2}{2}\right)} = \frac{3a^2\sqrt{7}}{4}.$$

Hence, $r = \frac{S}{p} = \frac{a\sqrt{7}}{2(\sqrt{2}+1)}$ and $\frac{R}{r} = \frac{8}{7}(2+\sqrt{2})$.

PROBLEMS TO BE SOLVED WITHOUT ASSISTANCE

I. Area of Triangles

220. Prove that the area of a triangle is equal to $\frac{2}{3} m_a m_b \sin\alpha$, where m_a and m_b are medians, and α is an angle between them.

221. In a triangle ABC, $AC = 3$ cm, $\angle A = 30°$, and the radius of the circumscribed circle equals 2 cm. Prove that the area of the triangle ABC is less than 3 cm^2.

222. Prove that $S \leqslant \frac{b^2+c^2}{4}$, where b and c are sides of a triangle, and S is its area.

223. The sides of a triangle are equal to 55 cm, 55 cm, and 66 cm. Find the area of the triangle whose vertices are the feet of the angle bisectors of the given triangle.

224. In a triangle ABC, $AB = 13$ cm, $BC = 15$ cm, and $AC = 14$ cm. Drawn in the triangle are the altitude BH, the angle bisector BD, and the median

BM. Find: (a) the area of the triangle BHD; (b) the area of the triangle BMD; (c) the area of the triangle BHM.

225. Taken on each median of a triangle is a point dividing the median in the ratio 5:1, reckoning from the vertex. Find the area of the triangle with vertices at these points if the area of the original triangle is equal to 64 cm^2.

226. A square is inscribed in a triangle with base a. Find the area of the triangle if it is known that the side of the square is larger than half the base of the triangle, and the area of the square is one-fourth of the area of the triangle.

227. Circumscribed about a triangle ABC with the angle $B = 60°$ is a circle whose radius is 4 cm. The diameter of the circle perpendicular to the side BC intersects AB at a point M such that $AM:BM = 2:3$. Find the area of the triangle.

228. Find the area of a right triangle with hypotenuse c if it is known that the sum of the sines of its acute angles is equal to q.

229. Find the area of a right triangle with an acute angle α if it is known that the distance from the vertex of the other acute angle to the centre of the inscribed circle is equal to m.

230. In an acute triangle ABC, $AB = c$, the median $BD = m$, and $\angle BDA = \beta$ $(\beta < 90°)$. Find the area of the triangle ABC.

231. A straight line is drawn through the vertex of the angle α at the base of an isosceles triangle at an angle β to the base $(\beta < \alpha)$. This line divides the triangle into two parts. Find the ratio of the areas of these parts.

232. A straight line is drawn through the midpoint of a side of a regular triangle. The line forms an acute angle α with this side. Find the ratio of the areas of the parts into which this line divides the triangle.

233. In a triangle ABC, $\angle A = \alpha$ and $\angle C = \gamma$. Drawn in it are the angle bisector BD, the altitude BH, and the median BM. Find: (a) the ratio of the area of the triangle BDM to the area of the triangle ABC; (b) the ratio of the area of the triangle BHM to the area of the triangle ABC; (c) the ratio of the area of the triangle BHD to the area of the triangle ABC.

234. Find the area of a triangle given its sides a and b, and the angle bisector $l_c = l$.

235. The median AD of a triangle ABC intersects the circle circumscribed about the triangle at the point E. Find the area of the triangle ABC if it is known that $\angle BAD = 60°$, $AB + AD = DE$, and $AE = 6$.

236. Prove that $S < \frac{1}{2}\pi R^2$, where S is the area of a triangle, and R the radius of the circle circumscribed about it.

237. One of the angles of a triangle is equal to 60°. The point of tangency of the inscribed circle divides the opposite side into two segments a and b. Find the area of the triangle.

238. Drawn from the point M situated on the side AB of a triangle ABC are lines $MQ \parallel AC$ and $MP \parallel BC$. Find the area of the triangle ABC if it is known that the area of the triangle BMQ is equal to S_1, and the area of the triangle AMP to S_2.

239. Lines are drawn through a point taken inside a triangle. These lines are parallel to its sides and divide the triangle into six parts among which there are three triangles with areas S_1, S_2, and S_3. Find the area of the original triangle.

240. A circle is inscribed in a triangle with sides 16 cm, 30 cm, and 34 cm. Find the area of the triangle whose vertices lie in the points of tangency.

241. A triangle ABC with side $AC = 20$ cm is inscribed in a circle. A tangent to the circle is drawn through the point B, the tangent being 25 cm and 16 cm distant from the points A and C, respectively. Find the area of the triangle ABC.

242. The perpendiculars MD, ME, and MF are dropped from a point M situated inside a triangle ABC to the sides AB, BC, and AC, respectively. Find

the ratio of the areas of the triangles ABC and DEF if it is known that $AB = c$, $BC = a$, $AC = b$, $ME = k$, $MF = m$, and $MD = n$.

243. The altitudes AD, BE, and CF are drawn in a triangle ABC. Find the ratio of the areas of the triangles DEF and ABC if the angles of the triangle ABC are α, β, and γ.

244. The chord AB subtends an arc of a circle whose length is one-third of the circumference of the circle. A point C is taken on this arc, and a point D on the chord AB. Find the area of the triangle ABC if it is known that $AD = 2$ cm, $BD = 1$ cm, and $CD = \sqrt{2}$ cm.

245. In a triangle ABC, the angle C is equal to 60° and the radius of the circumscribed circle is equal to $2\sqrt{3}$ cm. Taken on AB is a point D such that $AD:DB = 2:1$, and $CD = 2\sqrt{2}$ cm. Find the area of the triangle ABC.

246. The angle A of an isosceles triangle ABC ($AB = BC$) is equal to $\arcsin \frac{5}{13}$. A circle whose centre is $\frac{13}{24}$ cm distant from the vertex B touches the lateral sides AB at the point K and BC at the point P, and cuts off a line segment EF on the base AC. Find the area of the triangle EPC if it is known that $PC = \frac{6}{5}$ cm.

247. A circle is circumscribed about a triangle ABC. A tangent to the circle at the point B intersects the straight line AC at the point D (the point C lies between the points A and D). Find the area of the triangle BCD if it is known that $\angle BDC = \arccos \frac{21}{29}$, $BD = 29$ cm, and the distance from the centre of the circle to AC is equal to 10 cm.

II. Area of Quadrilaterals

248. Straight lines are drawn through the vertices of a quadrilateral parallel to its diagonals. Prove that the area of the parallelogram thus obtained is twice the area of the given quadrilateral.

249. The sides of a parallelogram are a and b, and the angle between them is α. Find the area of the quadrilateral formed by the bisectors of the interior angles of the parallelogram.

250. The median of an isosceles trapezoid is equal to a, and the diagonals are mutually perpendicular. Find the area of the trapezoid.

251. The perimeter of a trapezoid is 52 cm, and the smaller base is 1 cm. Find the area of the trapezoid if it is known that its diagonals are the bisectors of the obtuse angles.

252. Two circles of radii of 4 cm and 8 cm with centres at points O_1 and O_2 intersect at points C and D, AB being their common external tangent. Find the area of the quadrilateral O_1ABO_2 if it is known that the tangents drawn to the circles at C are mutually perpendicular.

253. Two equal circles of radius R with centres at points O_1 and O_2 touch each other externally. A straight line l intersects these circles at points A, B, C, and D so that $AB = BC = CD$. Find the area of the quadrilateral O_1ADO_2.

254. The sides of a triangle are equal to 20 cm, 34 cm, and 42 cm. The altitude lying inside the triangle is divided in the ratio 3:1, as measured from the vertex, and a straight line is drawn through the division point perpendicular to this altitude. Find the area of the trapezoid thus obtained.

255. The sides of a triangle are equal to 20 cm, 34 cm, and 42 cm. Find the area of the inscribed rectangle if it is known that its perimeter is equal to 45 cm.

256. The bases of a trapezoid are 62 cm and 20 cm, the nonparallel sides being 45 cm and 39 cm. Find the area of the trapezoid.

257. The bases of a trapezoid are 30 cm and 12 cm, the diagonals being 20 cm and 34 cm. Find the area of the trapezoid.

258. One of the bases of a trapezoid is 7 cm. The circle inscribed in the trapezoid divides one of the lateral sides into line segments, 4 cm and 9 cm long. Find the area of the trapezoid.

259. In a trapezoid $ABCD$, K is the midpoint of the base AD, M is the midpoint of the base BC, BK is the bisector of the angle ABC, and DM is the bisector of the angle ADC. Find the area of the trapezoid $ABCD$ if its perimeter is equal to 30 cm and $\angle BAD = 60°$.

260. In a convex quadrilateral $ABCD$, points E, F, P, and K are the midpoints of the sides AB, BC, CD, and AD, respectively. It is known that $EP = KF$. Find the area of the quadrilateral $ABCD$ if $AC = 15$ cm and $BD = 20$ cm.

261. Find the area of a parallelogram given its sides a and b $(a > b)$ and the angle α between its diagonals.

262. Find the area of a trapezoid with an acute angle α at the base if it is known that one of the bases of the trapezoid is a diameter of the circle of radius R circumscribed about the trapezoid.

263. A circle is inscribed in a trapezoid with acute angles α and β. Find the ratio of the area of the trapezoid to the area of the circle.

264. In a triangle ABC, $\angle A = \alpha$, $\angle B = \beta$, $\angle C = \gamma$, and the altitude $BD = H$. Constructed on BD as on the diameter is a circle intersecting the sides AB and BC at points E and F, respectively. Find the area of the quadrilateral $BFDE$.

265. A straight line l, parallel to the base AC of a triangle ABC, cuts off a triangle BED from this triangle. An arbitrary point M is taken on the side AC. Prove that the area of the quadrilateral $BEMD$ is the geometric mean between the area of the triangle ABC and the area of the triangle BED.

266. The diagonals of a trapezoid $ABCD$ $(AD \parallel BC)$ intersect at the point O. Find the area of the trapezoid if it is known that the area of the triangle AOD is equal to a^2, and the area of the triangle BOC to b^2.

267. In a rhombus $ABCD$, M, N, P, and Q are the midpoints of the sides AB, BC, CD, and AD, respectively. Find the area of the quadrilateral bounded by the straight lines AN, BP, DM, and CQ if the area of the rhombus is equal to 100 cm^2.

268. Two circles of radii a and b touch each other externally. Common external tangents are drawn to them. Find the area of the quadrilateral whose vertices are the points of tangency.

269. The diagonals of a quadrilateral $ABCD$ intersect at the point O. Find the area of the quadrilateral if it is known that the areas of the triangles AOB, BOC, and COD are equal to 12 cm^2, 18 cm^2, and 24 cm^2.

270. A circle touches the sides AB and AD of a rectangle $ABCD$, passes through the vertex C, and intersects the side DC at the point K. Find the area of the quadrilateral $ABKD$ if $AB = 9$ cm and $AD = 8$ cm.

271. A point M is taken inside a rectangle $ABCD$ so that $AM = \sqrt{2}$, $BM = 2$, and $CM = 6$. Find the area of the rectangle $ABCD$ if it is known that $AD = 2AB$.

III. Area of Polygons

272. Constructed externally on the legs AC and BC and on the hypotenuse AB of a right triangle ABC are squares $CMPA$, $BEFC$, and $ADKB$. Find the area of the hexagon $DKEFMP$ if $AB = c$ and $S_{\Delta ABC} = S$.

273. Constructed on the sides AC, BC, and AB of a triangle ABC are squares $CMPA$, $BEFC$, and $ADKB$. Find the area of the hexagon $DKEFMP$ if it is known that $AB = 13$ cm, $AC = 14$ cm, and $BC = 15$ cm.

274. A given square with side a is cut off at the corners so that a regular octagon is obtained. Find the area of the octagon.

275. Given a square with side a. Constructed externally on each side of the square is a trapezoid so that the upper bases of the trapezoids and their lateral sides form a regular dodecagon. Find the area of the dodecagon.

276. A circle is divided into eight parts by the points A, B, C, D, E, F, P, and K. It is known that $\cup AB = \cup CD = \cup EF = \cup PK$ and $\cup BC = \cup DE = \cup FP = \cup KA$; besides, $\cup AB = 2\cup BC$. Find the area of the octagon $ABCDEFPK$ if the area of the circle is equal to 289π cm^2.

277. Inscribed in a circle of radius R are a regular triangle and a square having a common vertex. Find the area of their intersection.

278. Each side of a triangle is divided into three parts in the ratios 3:2:3. Find the ratio of the area of a hexagon, whose vertices are the division points, to the area of the triangle.

279. The area of a quadrilateral $ABCD$ is equal to 12 cm^2. Taken on the sides AB, BC, CD, and DA are points F, K, M, and P, respectively, such that $AF:FB = 2:1$, $BK:KC = 1:3$, $CM:MD = 1:1$, and $DP:PA = 1:5$. Find the area of the hexagon $AFKCMP$.

IV. Area of Combined Figures

280. The sides of a triangle are 20 cm, 34 cm, and 42 cm. Find the ratio of the areas of the inscribed and circumscribed circles.

281. The side of a regular triangle is equal to a. Constructed on the side as on the diameter is a circle. Find the area of the part of the triangle lying outside the circle.

282. A circle is inscribed in a regular triangle. Another circle is described from one of the vertices of the triangle as centre, the radius of the second circle being equal to half the side of the triangle. The area of the intersection of the circles is what part of the area of the triangle?

283. Two circles of radii a and b $(a > b)$ touch each other externally. A common external tangent is drawn to them. Find: (a) the area of the curvilinear triangle thus obtained; (b) the area of the circle inscribed in this triangle.

284. The side of a regular triangle is equal to a. The centroid of the triangle serves as the centre of the circle of radius $\frac{a}{3}$. Find the area of the part of the triangle lying outside the circle.

285. Semicircles are constructed on each side as on the diameter inside a square with side a. Find the area of the rosette thus obtained.

286. Each of n equal circles touches two neighbouring ones. Find the area of the figure bounded by the nearest to each other arcs of these circles if it is known that the radius of the circle with which all the given circles have an internal contact is equal to R and that: (a) $n = 3$; (b) $n = 4$; (c) $n = 6$.

287. Two equal chords are drawn from a point taken on a circle of radius R, the angle between the chords being equal to α. Find the area of the portion of the circle enclosed between these chords.

288. Let $ABCDEK$ denote a regular hexagon with side a, and O its centre. Three circles are drawn: the first with centre at A passes through the points C and E, the second with centre at B passes through the points O and C, and the third with centre at K passes through the points O and E. Find the area of the figure bounded by these three circles and lying inside the hexagon.

289. Two circles of radii R and $2R$ are arranged so that the distance between their centres O_1 and O_2 is equal to $2R\sqrt{3}$. Drawn to them are common tangents intersecting at some point of the line segment O_1O_2. Find the area of the figure bounded by the segments of the tangents and the major arcs of the circles joining the points of tangency.

290. The base of a triangle is equal to a, the base angles being 15° and 45°. A circle is constructed with centre at the vertex of the triangle opposite to its

base and of radius equal to the altitude drawn from this vertex. Find the area of the portion of the circle lying inside the triangle.

291. Two circles of the same radius are arranged so that the distance between their centres is equal to the radius. Find the ratio of the area of the intersection of the circles to the area of the square inscribed in this intersection.

292. Given a semicircle with diameter AB, C being an arbitrary point belonging to the diameter AB. At the point C, a perpendicular CD is erected to the diameter to intersect the semicircle at the point D. On AC and CB as on diameters, two semicircles are constructed inside the given one. Prove that the area of the figure bounded by the three semicircles is equal to the area of the circle constructed on CD as on the diameter.

293. In a triangle ABC, $\angle A = \alpha$, $\angle B = \beta$, and $AC = b$. The altitudes AD and BE intersect at the point H. A circle is circumscribed about the triangle HDE. Find the area of the intersection of the circle and triangle.

294. Arranged inside a regular n-gon with side a are n equal circles so that each touches two others and a side of the n-gon. Find the area of the "star" formed at the centre of the n-gon.

295. Arranged inside a regular n-gon with side a are n equal circles so that each touches two adjacent sides of the n-gon and two other circles. Find the area of the "star" formed inside the n-gon.

V. Miscellaneous Problems

296. The area of a triangle is equal to 16 cm², the medians m_a and m_b are equal to 6 cm and 4 cm, respectively. Prove that these medians are mutually perpendicular.

297. An arbitrary point is taken inside a regular n-gon. Perpendiculars are dropped from this point to the sides or to their extensions. Prove that the sum of these perpendiculars is a constant quantity.

298. A straight line is drawn through the centroid of a regular triangle ABC parallel to the side AB. Taken on this line inside the triangle is an arbitrary point M, and from this point the perpendiculars MD, ME, and MF are dropped to the sides AB, AC, and BC, respectively. Prove that $MD = \frac{1}{2}(ME + MF)$.

299. Prove that $\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{1}{r}$, where h_1, h_2, and h_3 are the altitudes of a triangle, and r is the radius of the inscribed circle.

300. Let D be an interior point of the side AC of a triangle ABC, r_1 and r_2 the radii of the circles inscribed in the triangles ABD and BDC, respectively, and r the radius of the circle inscribed in the triangle ABC. Prove that $r < r_1 + r_2$.

301. The area of a convex quadrilateral $ABCD$ is equal to 3024 cm², and the diagonals to 144 cm and 42 cm. Find the length of the line segment joining the midpoints of the sides AB and CD.

302. The area of an isosceles triangle is equal to S, and the angle between the medians drawn to the lateral sides is equal to α. Find the base of the triangle.

303. The sides of a triangle are a and b, the angle between them being γ. Find: (a) the angle bisector l_c; (b) the altitude h_c.

304. The base of a triangle is equal to a, and the altitude to h. Find the sum of the lateral sides if it is known that the angle between them is α.

305. One of the angles of a triangle is equal to the difference between two others, the length of the smaller side equals 1 cm, and the sum of the areas of the squares constructed on two other sides is twice the area of the circle circumscribed about the triangle. Find the length of the larger side of the triangle.

306. Known in a triangle are two sides a and b $(a > b)$ and the area S. Find the angle between the altitude and median drawn from the common vertex of the two given sides.

307. Knowing the area S and the angles α, β, and γ of a triangle, find the length of the altitude drawn from the vertex of the angle α.

308. Inscribed in a triangle ABC is a circle touching the side AB at the point M, and the side AC at the point N. Find the angle BAC and the radius of the inscribed circle if $AM = 1$ cm, $BM = 6$ cm, and $CN = 7$ cm.

309. The area of a rectangle $ABCD$ is equal to 48 cm^2, and the diagonal to 10 cm. The point O is 13 cm distant from the vertices B and D. Find the distance from the point O to the most remote vertex of the rectangle.

310. The sides a, b, and c of a triangle form an increasing arithmetic progression. Prove that $ac = 6Rr$, where R and r are the respective radii of the circumscribed and inscribed circles.

311. The bases of a trapezoid are a and b. Find the length of the line segment which divides the trapezoid into two equal parts, is parallel to the bases and enclosed between the lateral sides.

312. In a trapezoid $ABCD$, the base $AD = BC = 12$ cm. A point M is taken on the extension of BC beyond the point C so that the area of the triangle cut off by the straight line AM from the trapezoid $ABCD$ is equal to one-third of the area of the trapezoid. Find the length of the line segment CM.

313. The altitudes AD and CE of an acute triangle ABC are dropped from the vertices A and C. It is known that the area of the triangle ABC is equal to 18 cm^2, and the area of the triangle BDE is equal to 2 cm^2, the length of the line segment DE being equal to $2\sqrt{2}$ cm. Compute the radius of the circle circumscribed about the triangle ABC.

314. The altitudes AD and CE of an acute triangle ABC are dropped from the vertices A and C. It is known that the area of the triangle ABC is equal to 64 cm^2, and the area of the triangle BDE is equal to 16 cm^2. Find the length of the line segment DE if the radius of the circle circumscribed about the triangle ABC is equal to $16\sqrt{3}$ cm.

315. Taken on the sides AB and AC of the triangle ABC whose area is equal to 6 cm^2 are respectively points K and M such that $AK{:}BK = 2{:}3$ and $AM{:}CM = 5{:}3$. The straight lines CK and BM intersect at the point P. Find AB if the distance from the point P to the line AB is equal to 1.5 cm.

316. The angle bisector AD of an isosceles triangle ABC $(AB = BC)$ is drawn. Find AC if $S_{\Delta ABD} = S_1$ and $S_{\Delta ADC} = S_2$.

317. In a triangle ABC, H is the orthocentre. Find the line segment AH if $AB = 13$ cm, $BC = 14$ cm, and $AC = 15$ cm.

318. The centre of the circle inscribed in a triangle is joined to the vertices of the triangle. As a result, three triangles are obtained whose areas are 4 cm^2, 13 cm^2, and 15 cm^2. Find the sides of the original triangle.

319. In a triangle ABC it is known that $BC{:}AC = 3$ and $\angle C = \gamma$. Taken on AB are points D and K such that $\angle ACD = \angle DCK = \angle KCB$. Find the ratio $CD{:}CK$.

320. The median BD of a triangle ABC is drawn. Find the ratio of the radius of the circle circumscribed about the triangle ABD to the radius of the circle inscribed in the triangle ABC if $AB = 2$, $AC = 6$, and $\angle BAC = 60°$.

321. In a triangle ABC it is known that $AC{:}BC = 1{:}3$ and $\angle ACB = \arctan \dfrac{\sqrt{5}}{2}$. Taken on the side AC is a point D such that $AC = CD$. Find the ratio of the area of the circle circumscribed about the triangle ACD to the area of the circle inscribed in the triangle ABD.

SEC. 5. GEOMETRICAL TRANSFORMATIONS

Let us consider the notions of *central symmetry* and also a *composition of central symmetries* by way of examples.

Example 1. Through a point lying inside a circle draw a chord to be bisected by the given point.

Solution. We construct a circle symmetric to the given circle about the given point. The desired chord will be a common chord of these circles.

Example 2. Construct a pentagon given the midpoints of its sides.

Solution. Let us denote the midpoints of the sides of the pentagon to be constructed by M, N, P, Q, and K. We take an arbitrary point A and consider the composition of central symmetries $Z_K \circ Z_Q \circ Z_P \circ Z_N \circ Z_M$. What is done by this composition with the point A? If the composition is denoted by δ, then $\delta(A) = A$ (Fig. 72). Let $Z_P \circ Z_N \circ Z_M(A) = Z_S(A)$, then $Z_K \circ Z_Q \circ Z_S(A) = A$. Construct the parallelograms $MNPS$ and $SQKA$, and the further construction of the desired pentagon becomes obvious.

Let us consider a couple of examples of application of *axial symmetry*.

Example 3. Given two circles ω_1 and ω_2 and a straight line l. Construct an equilateral triangle such that two of its vertices belong to the given circles, the third vertex belonging to the given straight line (Fig. 73).

Solution. Suppose that $\triangle ABC$ is the desired triangle. Since the altitude AD of the equilateral triangle ABC belongs to the line l, the points B and C are symmetric about this line and lie on the given circles ω_1 and ω_2.

Since the point C belongs to the circle ω_2 and is symmetric to the point B belonging to the circle ω_1 about the line l, the point C also belongs to the image of the circle ω_1 in the symmetry about the line l. Consequently, C is a common point of the circle ω_2 and the image of the circle ω_1 in the symmetry S_l. Thus, having constructed the

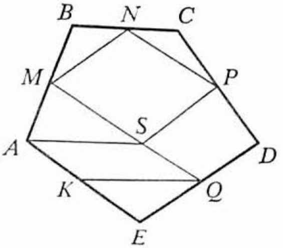


Fig. 72

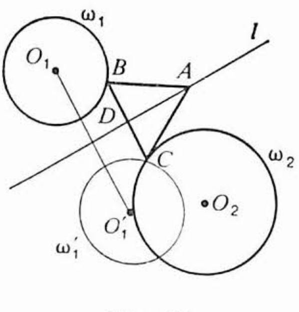


Fig. 73

circle ω_1' symmetric to the circle ω_1 about the line l, we find the point C.

We then construct the point B as the image of the point C in the symmetry S_l, and further the point A.

The constructions to be carried out when solving this problem are fulfilled in the following order: (1) construct the image of the circle ω_1 in the symmetry S_l; (2) find the points of intersection of the circles ω_1' and ω_2; (3) on the circle ω_1 find the pre-images of the points of intersection of the circles ω_1' and ω_2; (4) construct the equilateral triangle ABC whose vertex A belongs to the line l.

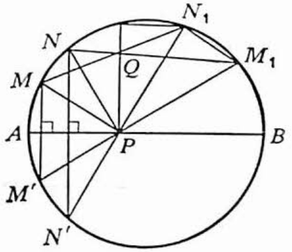


Fig. 74

If the circles ω_1' and ω_2 intersect, then the problem has four solutions. If the circles ω_1' and ω_2 touch each other, then the problem has two solutions. If the circle ω_1' coincides with the circle ω_2, then the problem has infinitely many solutions. If the circles ω_1' and ω_2 have no common points, then the problem has no solution.

Example 4. Given on the diameter AB of a semicircle is a point P, and on its semicircumference, points M, M_1 and N, N_1 such that $\angle MPA = \angle M_1PB$ and $\angle NPA = \angle N_1PB$. Prove that the point Q of intersection of the chords MN_1 and M_1N belongs to the perpendicular drawn to the diameter AB through the point P (Fig. 74).

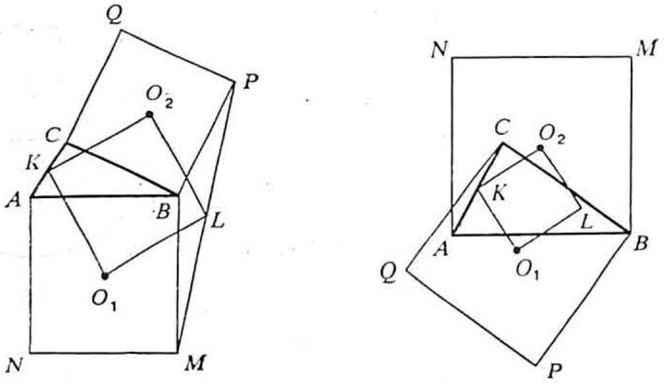
Solution. We construct the points M' and N' symmetric to the points M and N about the straight line AB. Then it is given that the points M', P, and M_1 lie in the same straight line, and, analogously, the points N', P, and N_1 belong to one straight line. A circle can be described about the quadrilateral PQN_1M_1, and, consequently, $\angle QPN_1 = \angle N_1M_1Q$ (or their sum is equal to 180°). We then consider the quadrilateral $PQNM$ possessing the same property: $\angle NPQ = \angle NMQ$. From the congruence of the angles NMQ and N_1M_1Q it follows that $\angle NPQ = \angle N_1PQ$ and $PQ \perp AB$.

The following two examples illustrate how *rotation* is applied.

Example 5. Construct an equilateral triangle one vertex of which coincides with a given point A, and two others belong to two given circles.

Solution. We construct the image of one of the circles when rotating it through an angle of 60° with centre of rotation at the point A. The point of intersection of the second of the two given circles and the constructed circle is the second vertex of the triangle.

Example 6. Constructed on the sides AB and BC of the triangle ABC as on bases are the equally oriented squares $ABMN$ and $BCQP$. Let us denote their centres by O_1 and O_2, the midpoint of the side AC by K, and the midpoint of the line segment MP by L. Prove that the quadrilateral O_1LO_2K is a square.



(a) (b)

Fig. 75

Solution. Consider the case when the given squares are constructed externally with respect to the triangle ABC. Note that the composition $R_{O_2}^{270°} \circ R_{O_1}^{270°}$ carries the point A into the point C, therefore $R_{O_2}^{270°} \circ R_{O_1}^{270°} = R_K^{180°}$. Hence it follows that the triangle O_1O_2K is right and isosceles.

Similarly, $R_{O_2}^{90°} \circ R_{O_1}^{90°} = R_L^{180°}$, therefore O_1O_2L is also a right and isosceles triangle ($\angle L = 90°$). Consequently, O_1LO_2K is a square (Fig. 75*a*).

In the case when the squares are arranged internally, the problem is solved in a similar way (Fig. 75*b*).

Let us consider a number of examples in which *translation* is applied.

Example 7. Two parallel lines p and q are intersected by a third line s. Construct an equilateral triangle with a given side so that its vertices belong to the lines p, q, and s (Fig. 76).

Solution. From an arbitrary point A_1 of the line q with radius equal to the length of the given segment we describe a circle and find the point C_1 at which it intersects the line p; we construct an equi-

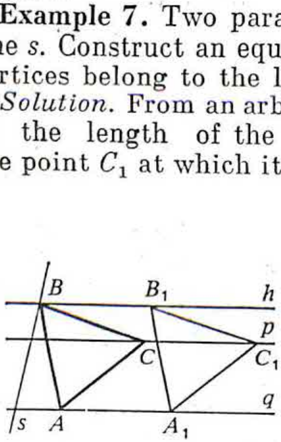


Fig. 76

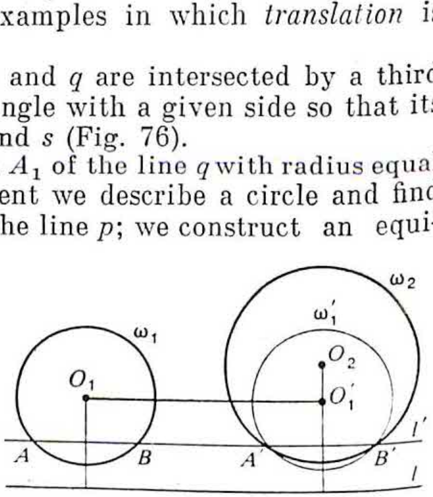


Fig. 77

lateral triangle $A_1B_1C_1$. Through the point B_1 we draw a straight line h parallel to p and denote the point of intersection of the lines h and s by B. We then carry out the translation $\mathbf{v}$ of the triangle $A_1B_1C_1$, where $\mathbf{v} = \overrightarrow{B_1B}$. The problem can have two solutions, one solution, and no solution.

Example 8. Given two circles ω_1 and ω_2 and a straight line l. Draw a straight line parallel to the line l on which the circles ω_1 and ω_2 cut equal chords (Fig. 77).

Solution. Let the line l' cut equal chords AB and $A'B'$ on the given circles. In this case, the points A and A', B and B' may be regarded as corresponding in the translation $T_{\overrightarrow{O_1O_1'}}$, where $\overrightarrow{O_1O_1'}$ is a vector whose initial point is O_1, that is, the centre of the circle ω_1, then O_1' is the centre of the circle ω_1'.

Since the point A' is the image of the point A belonging to the circle ω_1, the point A' belongs to the image of the circle ω_1'. Consequently, the point A' is a common point of the circles ω_2 and ω_1' in the translation $T_{\overrightarrow{O_1O_1'}}$.

On constructing the point A', we find its preimage on the circle ω_1, then AA' is just the desired line l'.

If the circle ω_1' coincides with the circle ω_2, the problem has infinitely many solutions. In all other cases, the problem has no more than one solution.

How *homothetic transformation* is applied is clear from the following examples.

Example 9. Drawn in the trapezoid $ABCD$ are the diagonals AC and BD intersecting at the point M (AB and CD being the bases of the trapezoid). Prove that the areas of the triangles ABM and CDM, respectively equal to S_2 and S_1, and the area S of the trapezoid are related as follows: $\sqrt{S_1} + \sqrt{S_2} = \sqrt{S}$ (Fig. 78).

Solution. Let N be the point of intersection of the line AB and the straight line passing through the point C parallel to DB. The area of the triangle ACN is equal to the area S of the given trapezoid. Draw BF parallel to AC. The area of the triangle BFN is equal to the area S_1 of the triangle DMC. The triangles AMB and BFN are homothetic to the triangle ACN with the ratios of similitude k_1 and k_2, where $k_1 + k_2 = 1$. But $k_1 = \dfrac{\sqrt{S_1}}{\sqrt{S}}$ and $k_2 = \dfrac{\sqrt{S_2}}{\sqrt{S}}$, consequently, $\sqrt{S_1} + \sqrt{S_2} = \sqrt{S}$.

Example 10. Inscribed in the triangle ABC is a circle touching the line AB at the point M. Let the point M_1 lie diametrically opposite to the point M on the inscribed circle. Prove that the line CM_1 intersects the line AB at a point C_1 such that $AC + AC_1 = BC + BC_1$ (Fig. 79).

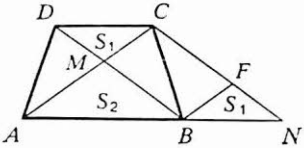


Fig. 78

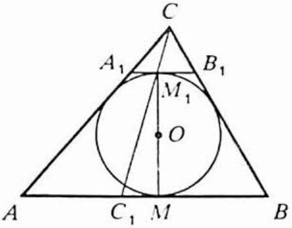


Fig. 79

Solution. We construct a tangent to the circle at the point M_1 intersecting AC at the point A_1 and BC at the point B_1. Then it is clear that $CA_1 + A_1M_1 = CB_1 + B_1M_1$. Further, we take advantage of the fact that the triangles ABC and $A_1B_1C_1$ are homothetic since the lines AB and A_1B_1 are perpendicular to the diameter MM_1, and, therefore, $AB \parallel A_1B_1$.

PROBLEMS TO BE SOLVED WITHOUT ASSISTANCE

I. Symmetry with Respect to a Point

322. Given a straight line, a line segment, and a point O. Construct a line segment so that its end points belong to the given line and line segment, the point O being its middle.

323. Drawn in a triangle ABC are three medians, i.e. AA_1, BB_1, and CC_1, intersecting at a common point M. Points P, Q, and R are the midpoints of the line segments AM, BM, and CM. Prove that $\triangle A_1B_1C_1 = \triangle PQR$.

324. Construct a triangle given two sides and a median drawn to the third side. Within what limits can the length of the median vary if the lengths of the sides of the triangle are equal to a and b?

325. Points M, N, and K are the midpoints of the line segments whose one end is a vertex of a triangle ABC, and the other is the point of intersection of its medians (or the median point). Prove that the triangle whose vertices are the points of intersection of the lines, parallel to the corresponding sides of the triangle ABC and containing the points M, N, and K, is congruent to the triangle ABC.

326. Given two circles and a point P. Construct a parallelogram so that its vertices belong to the given circles, and the point P is the intersection of the diagonals of the parallelogram.

327. A straight line containing the point of intersection of the diagonals of a parallelogram $ABCD$ cuts off line segments BE and DF on its sides. Prove that these segments are congruent.

328. Divide a parallelogram into two equivalent parts.

329. Two equal chords BA and CD are drawn from the ends of the diameter BC of a circle with centre O so that BA and CD do not intersect and lie on different sides from BC. Prove that OA and OD belong to one line and that $DO = OA$.

330. Circumscribed about a circle is a hexagon with parallel opposite sides. Prove that the opposite sides of this hexagon are equal.

331. The opposite sides of a convex hexagon $ABCDEF$ are pairwise parallel and equal. The area of the triangle ACE is what part of the area of the hexagon?

332. Given points A and B on a circle and a point M on a straight line l. On the circle find a point X such that the lines AX and BX intersect the line l at the points equidistant from the point M.

333. Through the point M not belonging to the sides of an angle ABC draw a secant such that a triangle of the least area is obtained.

334. Circumscribed about a circle is an octagon whose opposite sides are pairwise parallel. Prove that the opposite sides of the octagon are pairwise equal.

335. Given a triangle ABC and a point X. Construct the parallelogram $BXCY$, and then another parallelogram $YXAZ$. Prove that there is a homothetic transformation carrying the point X into the point Z and find its ratio of similitude and centre.

336. In a given quadrilateral inscribe a parallelogram, provided that two vertices of the parallelogram are fixed and belong to: (a) the opposite sides; (b) the adjacent sides of the quadrilateral.

337. The median CM of a triangle ABC forms angles α and β with the sides AC and BC, respectively. Which of these angles is larger if $AC < BC$?

II. Symmetry About a Straight Line

338. Construct a pentagon having: (a) one axis of symmetry; (b) more than one axis of symmetry.

339. Through a given point draw a straight line intersecting two given lines at equal angles.

340. Construct a triangle given a side, the difference between two other sides, and the angle included between the first side and the larger of the two other sides.

341. Construct a triangle given two sides and the difference between the opposite angles.

342. A point M is given inside an acute angle. Construct the triangle MAB having the least perimeter, its vertices A and B lying on the sides of the angle.

343. Construct a convex quadrilateral $ABCD$ having only one axis of symmetry—the line BD.

344. Is it possible to construct a pentagon such that its diagonal lies on its axis of symmetry. Give grounds for your answer.

345. Prove that in a convex polygon with an odd number of vertices and having axes of symmetry, none of its diagonals can lie on an axis of symmetry.

346. Construct a triangle given an angle, an adjacent side, and the difference between two other sides.

347. Construct a triangle given a nonzero difference between two of its angles and the lengths of the opposite sides.

348. Given two concentric circles. Construct a rhombus, different from a square, in which: (a) two vertices belong to one circle, and two others to the other circle; (b) three vertices belong to one circle, and one vertex to the other circle.

349. Construct a triangle ABC given three middle perpendiculars p, q, and r to its sides.

350. In a given circle inscribe a triangle whose sides are parallel to three given lines.

351. Circumscribed about a triangle ABC is a circle intersecting the bisector of the angle C at a point M. A perpendicular HD is dropped from the orthocentre H of the triangle to the angle bisector so that the point D belongs to l_C. Prove that $CD:CM = \cos C$.

352. In a circle with centre O, a quadrilateral $ABCD$ is inscribed and the rays OM, ON, OP, and OQ are constructed, where M, N, P, and Q are the midpoints of the chords AB, BC, CD, and DA. Prove that $\angle MON + \angle COD = 180°$, or $\angle MON = \angle POQ$.

353. A quadrilateral $ABCD$ is circumscribed about a circle with centre O. Prove that $\angle AOB + \angle COD = 180°$.

354. In a given circle inscribe a pentagon whose sides are parallel to five given lines.

355. A ball lies on a billiard table of rectangular shape. In what direction is it necessary to hit the ball so that, on having been reflected from all the cushions, it rolls through its initial position?

356. Prove that the point of intersection of the straight lines containing the lateral sides of an isosceles trapezoid, the point of intersection of its diagonals, and the midpoints of the bases of the trapezoid belong to the same straight line.

357. Prove that a straight line containing the midpoints of two parallel chords of a circle passes through its centre.

358. A circle F_1 intersects two concentric circles F_2 and F_3 at points A, B and C, D, respectively. Prove that the chords AB and CD are parallel.

359. Three equal circles have a common point. Prove that the circle drawn through the second points of intersection of the given three circles is equal to the given circles.

360. Given in a plane are four equal circles passing through a common point and intersecting for the second time at six points. Prove that four circles passing through each three of these six points, one taken on each of the given circles, intersect at a common point.

361. Given in a plane are a straight line and a point not belonging to this line. Find the locus of the centre of regular triangles, one vertex of which is situated at the given point, and the other on the given line.

362. Given in a plane are a straight line and a point not belonging to this line. Find the locus of the third vertices of regular triangles, one vertex of which is found at the given point, and the other on the given line.

III. Rotation

363. Construct the square $ABCD$ given its centre O and two points M and N which belong respectively to the lines AB and BC, $OM \neq ON$.

364. Construct an equilateral triangle such that one of its vertices coincides with a given point O, and two others belong to two given circles.

365. Through a point given inside a circle draw a chord of a given length.

366. Given on the sides BC, CA, and AB of an equilateral triangle ABC are points M, N, and P, respectively. It is known that $BM:MC = CN:NA = AP:PB = k$. (a) Prove that MNP is an equilateral triangle. (b) Compute MN if $BC = a$ and $k = 2$.

367. Given on the sides BC, CD, DA, and AB of a square $ABCD$ are points P, Q, R, and S, respectively. It is known that $BP:PC = CQ:QD = DR:RA = AS:SB = h$. (a) Prove that $PQRS$ is a square. (b) Compute PQ if $AB = a$ and $h = 3$.

368. Constructed on the sides AB and BC of a triangle ABC as on bases are equally oriented squares $ABMN$ and $BCOP$. Let us denote their centres by O_1 and O_2, the midpoint of the side AC by K, the midpoint of the line segment MP by L. Prove that the quadrilateral O_1LO_2K is a square.

369. Equilateral triangles ACB_1 and BCA_1 are constructed externally on the sides AC and BC of a triangle ABC. Find the angles of the triangle MA_1O, where M is the midpoint of the side AB, and O is the centre of the triangle ACB_1.

370. Laid off on the extensions of the sides of a right triangle ABC are line segments AD and AE respectively equal to the legs AB and AC of the triangle ABC. Prove that the straight line containing the median AM of the triangle ABC is perpendicular to the line segment DE.

371. Given a square $ABCD$. Drawn through the centre of this square are two mutually perpendicular lines, different from the lines AC and BD. Prove

that the figures which are the intersections of these lines and the square are congruent.

372. Two straight lines, forming an angle of 60° between them, are drawn through the centre O of a regular triangle ABC. Prove that the segments of these lines enclosed inside the triangle are equal.

373. Construct an equilateral triangle such that one of its vertices is the point P, the other belongs to the line a, and the third to the line b.

374. Constructed externally on the sides AB and AC of a triangle ABC are squares $ABNM$ and $ACQP$. Prove that MC is perpendicular to BP.

375. Given two equally oriented squares $MPOR$ and $MUVW$. Prove that the line segments PU and RW are equal and mutually perpendicular.

376. Constructed on the sides AB and BC of a triangle ABC are squares with centres D and E such that the points C and D are situated on one side from AB, while the points A and E on different sides from BC. Prove that the angle between the lines AC and DE is equal to 45°.

377. Construct the square $ABCD$ given its centre O and two points M and N belonging to the lines BC and CD, $OM \neq ON$.

IV. Translation

378. Given four distinct points A, B, C, and D. Draw four parallel lines a, b, c, and d, respectively, through them so that the width of the strip between the lines a and b is equal to the width of the strip between the lines c and d.

379. Construct a trapezoid given its diagonals, the angle between them, and one of its sides.

380. Prove that if a straight line passing through the midpoints of the bases of a trapezoid forms equal angles with the lines containing its lateral sides, then the trapezoid is isosceles.

381. Two equal circles touch each other externally at the point K. A secant parallel to the line of centres intersects the circles at points A, B, C, and D. Prove that the size of the angle AKC is independent of the choice of the secant.

382. Determine the area of a trapezoid all sides of which are known.

383. Given on a circle with centre O are points A, B, and C such that $\angle AOB = \angle BOC = 60°$. Prove that the distance from the point B to an arbitrary diameter of the circle is equal to either the sum or the absolute value of the difference of the distances from the points A and C to this diameter.

384. Through the point M, lying outside the circle ω, draw a straight line m intersecting ω at two points A and B such that $AB = BM$.

385. Four equal circles ω_1, ω_2, ω_3, and ω_4 pass through the point M and, for the second time, intersect at six points: A_{12} is the point of intersection of ω_1 and ω_2, $A_{23}-\omega_2$ and ω_3, $\ldots$, $A_{43}-\omega_4$ and ω_3. Prove that the segments $A_{12}A_{43}$, $A_{23}A_{14}$, and $A_{13}A_{24}$ have a common midpoint.

386. The straight lines containing the lateral sides of a trapezoid are mutually perpendicular. Prove that the length of the line segment whose end points are represented by the midpoints of the bases of the trapezoid is equal to half the difference between the lengths of the bases.

387. The sum of the lengths of the bases of a trapezoid is equal to 21 cm, and the lengths of the diagonals are equal to 13 cm and 20 cm. Compute the area of the trapezoid.

388. The distance between the centres of two intersecting circles of equal radii equals d. A straight line, parallel to the line of centres, intersects the first circle at points A and B, the second at points C and D. Find the length of the segment AC (Fig. 80).

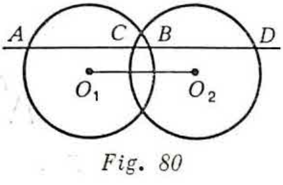


Fig. 80

389. Construct the quadrilateral $ABCD$ given

the length of its sides and the length of the segment MN joining the midpoints of the sides AB and DC.

390. The diagonals of a trapezoid whose bases are a and b are mutually perpendicular. What values can the altitude of the trapezoid take on?

V. Homothetic Transformation

391. Prove that in an arbitrary triangle ABC, the point M of intersection of medians, the point H of intersection of altitudes, and the centre O of the circumscribed circle belong to one and the same line (Euler's line) and $\frac{OM}{MH} = \frac{1}{2}$.

392. Given an angle ABC and a point M inside this angle. Draw a straight line through the point M so that its segment enclosed inside the angle ABC is divided by the point M in the ratio 1:2.

393. Drawn through the point M of contact of the circles R and S are the secants h and l intersecting the circle R at points A and B (in addition to the point M), and the circle S at points C and D. Prove that the lines AB and CD are parallel.

394. Prove that if an arbitrary straight line is drawn through the point of tangency of two circles, then it will intersect the circles for the second time at points such that the radii drawn to these points will be parallel.

395. Given three parallel, pairwise not equal line segments MN, PQ, and RS, the rays MN, PQ, and RS being in the same direction. Prove that three points of intersection of three pairs of lines, i.e. MP and NQ, MR and NS, PR and QS, belong to one straight line; that three points of intersection of three pairs of lines, i.e. MQ and NP, QR and PS, MR and NS, also belong to one straight line (Fig. 81).

396. Two circles touch each other internally at a point A. A secant a intersects the circles at points M, N, P, and Q situated in consecutive order (Fig. 82). Prove that $\angle MAN = \angle PAQ$.

397. The lengths of the line segments one end point of which is a common point and the other a point of the line are divided in the same ratio. Prove that the division points belong to one and the same line.

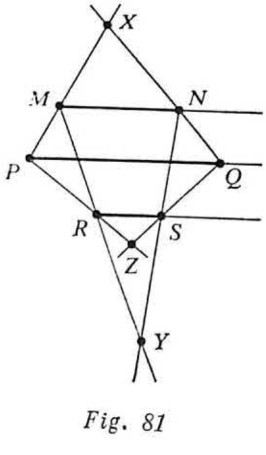


Fig. 81

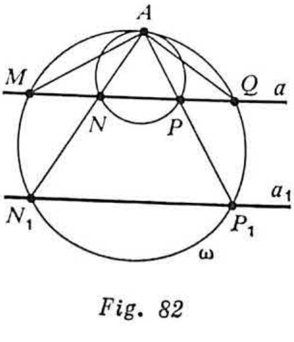


Fig. 82

SEC. 6. VECTORS

Addition of Vectors. The sum of the vectors **a** and **b** with coordinates a_1, a_2 and b_1, b_2 is defined as the vector **c** with coordinates $a_1 + b_1$, $a_2 + b_2$, that is, $\mathbf{a}(a_1, a_2) + \mathbf{b}(b_1, b_2) = \mathbf{c}(a_1 + b_1, a_2 + b_2)$.

For any vectors $\mathbf{a}(a_1, a_2)$, $\mathbf{b}(b_1, b_2)$, $\mathbf{c}(c_1, c_2)$ the equalities $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ are fulfilled.

Triangle Rule. For any three points A, B, and C, the equality $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ holds true.

Parallelogram Rule. If $ABCD$ is a parallelogram, then $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$.

Multiplication of a Vector by a Number. The product of a vector $(\mathbf{a}_1, \mathbf{a}_2)$ by a number λ is defined as the vector $(\lambda\mathbf{a}_1, \lambda\mathbf{a}_2)$, i.e. $(\mathbf{a}_1, \mathbf{a}_2)\lambda = (\lambda\mathbf{a}_1, \lambda\mathbf{a}_2)$. By definition, $(\mathbf{a}_1, \mathbf{a}_2)\lambda = \lambda(\mathbf{a}_1, \mathbf{a}_2)$.

For any vector **a** and numbers λ and μ, $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$.

For any vectors **a** and **b** and a number λ, $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$; $|\lambda\mathbf{a}| = |\lambda|\,|\mathbf{a}|$.

The direction of the vector $\lambda\mathbf{a}$ for $\mathbf{a} \neq \mathbf{0}$ coincides with that of the vector **a** if $\lambda > 0$, and is opposite to the direction of the vector **a** if $\lambda < 0$.

Two nonzero vectors are said to be *collinear* if they lie on one line or on parallel lines.

If $\mathbf{a}(a_1, a_2) \parallel \mathbf{b}(b_1, b_2)$, then $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. The converse is also true.

A *unit vector* is a vector of unit length. Unit vectors having directions of the positive coordinate semi-axes are called *basis vectors* and denoted as $\mathbf{e}_1(1, 0)$ on the x-axis and as $\mathbf{e}_2(0, 1)$ on the y-axis.

Any vector $\mathbf{a}(a_1, a_2)$ is representable in the form $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$.

Scalar Product of Two Vectors. The scalar product of a vector $\mathbf{a}(a_1, a_2)$ by a vector $\mathbf{b}(b_1, b_2)$ is defined as the number $a_1b_1 + a_2b_2$.

$\mathbf{a}\cdot\mathbf{a} = \mathbf{a}^2$, $\mathbf{a}^2 = |\mathbf{a}|^2$.

For any three vectors $\mathbf{a}(a_1, a_2)$, $\mathbf{b}(b_1, b_2)$, and $\mathbf{c}(c_1, c_2)$, $(\mathbf{a} + \mathbf{b})\mathbf{c} = \mathbf{ac} + \mathbf{bc}$.

The *angle between two nonzero vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ is defined as the angle BAC. The angle between two vectors **a** and **b** is defined as the angle between the vectors equal to them and having a common origin. The angle between two vectors in the same direction is regarded to be equal to zero.

The scalar product of two vectors is equal to the product of their absolute values by the cosine of the angle between them.

If $\mathbf{a} \perp \mathbf{b}$, then $\mathbf{a}\cdot\mathbf{b} = 0$, and if $\mathbf{a}\cdot\mathbf{b} = 0$, where $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, then $\mathbf{a} \perp \mathbf{b}$.

Required to be proved (using geometrical language)	Sufficient to prove (using vector language)
(1) $a \parallel b$	$\overrightarrow{AB}=k\overrightarrow{CD}$, where the line segments AB and CD belong to the lines a and b, respectively, and k is a number. Depending on the choice of AB and CD, various vector relationships occur, of which suitable ones are chosen.
(2) The points A, B, and C belong to the line a	(a) ascertain the validity of one of the following equalities: $\overrightarrow{AB}=k\overrightarrow{BC}$ or $\overrightarrow{AC}=k\overrightarrow{BC}$ or $\overrightarrow{AC}=k\overrightarrow{AB}$; (b) prove the equality $\overrightarrow{QC}=p\overrightarrow{QA}+q\overrightarrow{QB}$, where $p+q=1$, and Q is an arbitrary point; (c) prove the equality $\alpha\overrightarrow{OA}+\beta\overrightarrow{OB}+\gamma\overrightarrow{AC}=0$, where $\alpha+\beta+\gamma=0$, and Q is an arbitrary point.
(3) The point C belongs to the line segment AB, where $AC:AB=m:n$ (division of a line segment in a given ratio)	$\overrightarrow{AC}=\frac{m}{n}\overrightarrow{CB}$ or $\overrightarrow{QC}=\frac{n}{m+n}\overrightarrow{QA}+\frac{m}{m+n}\overrightarrow{QB}$ for a certain point Q.
(4) $a \perp b$	$\overrightarrow{AB}\cdot\overrightarrow{CD}=0$, where the points A and B belong to the line a, and the points C and D to the line b.
(5) Compute the length of a line segment	(a) choose two noncollinear basis vectors (or three noncoplanar) whose lengths and the angle between them are known; (b) decompose along them the vector whose length is being computed; (c) find the scalar square of this vector, using the formula $\mathbf{a}^2=\|\mathbf{a}\|^2$.
(6) Compute the size of an angle	(a) choose two noncollinear basis vectors for which the ratio of their lengths and the angle between them are known; (b) choose the vectors specifying the desired angle and decompose them along the basis vectors; (c) compute $\cos\angle(\mathbf{a}, \mathbf{b})=\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}$.

A body of vector algebra made it possible to create a specific method of solving various geometrical problems. However, it should be borne in mind that this method is not universal, it may turn out to be inapplicable for solving certain problems or not successful. The following table gives examples of use of the vector language for formulating and proving certain geometrical statements or computing geometrical quantities.

The reader may see concrete applications in the first three cases when considering affine problems, and in the last three cases when considering metric problems.

I. Affine Problems

Let us single out several types of affine problems which are appropriately solved by using vectors. (We mean only such problems whose formulation contains no concepts of vector algebra.)

The *first type* comprises problems involving the proof of the parallelism of certain line segments and straight lines. To solve problems of this type, it is necessary to prove the collinearity of vectors represented by given line segments, that is, to prove that $\mathbf{a} = k\mathbf{b}$, where k is a number. Consider a couple of examples to see how problems of the first type are solved.

Example 1. Given in the plane are a quadrilateral $ABCD$ and a point M. Prove that the points, symmetric to the point M with respect to the midpoints of the sides of this quadrilateral, are the vertices of a parallelogram.

Solution. Let $ABCD$ denote the given quadrilateral (Fig. 83), and let N, P, Q, and R be the points symmetric to the point M with respect to the midpoints of the line segments AB, BC, CD, and DA.

By the parallelogram rule, we have: $\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{MB}$, $\overrightarrow{MP} = \overrightarrow{MB} + \overrightarrow{MC}$, $\overrightarrow{MQ} = \overrightarrow{MC} + \overrightarrow{MD}$, and $\overrightarrow{MR} = \overrightarrow{MD} + \overrightarrow{MA}$.

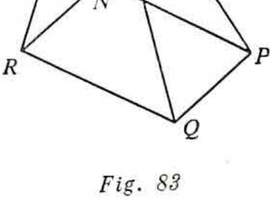


Fig. 83

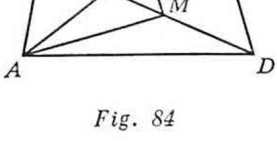


Fig. 84

By the definition of the difference between vectors, $\overrightarrow{NR} = \overrightarrow{MR} - \overrightarrow{MN}$ and $\overrightarrow{PQ} = \overrightarrow{MQ} - \overrightarrow{MP}$.

Since $\overrightarrow{NR} - \overrightarrow{PQ} = (\overrightarrow{MR} - \overrightarrow{MN}) - (\overrightarrow{MQ} - \overrightarrow{MP})$, using the initial equalities, we make sure that $\overrightarrow{NR} - \overrightarrow{PQ} = 0$, that is, $\overrightarrow{NR} = \overrightarrow{PQ}$. Similarly, it is proved that $\overrightarrow{NP} = \overrightarrow{RQ}$. Consequently, $\overrightarrow{NR} = \overrightarrow{PQ}$ and $\overrightarrow{NP} = \overrightarrow{RQ}$, and this means that the quadrilateral $NPQR$ is a parallelogram.

Example 2. Given a quadrilateral $ABCD$. The straight line drawn through the vertex A parallel to the side BC intersects the diagonal BD at the point M, and the straight line drawn through the vertex B parallel to the side AD intersects the diagonal AC at the point N. Prove that $MN \parallel DC$ (Fig. 84).

Solution. To solve the problem, it is sufficient to prove that the vectors $\overrightarrow{DC}$ and $\overrightarrow{MN}$ are collinear, that is, we have to prove that $\overrightarrow{DC} = k\overrightarrow{MN}$, where k is some number. To make sure that the vectors $\overrightarrow{DC}$ and $\overrightarrow{MN}$ are collinear, let us express each of them in terms of other vectors. Thus, the vector $\overrightarrow{DC}$ is expressed in terms of the vectors $\overrightarrow{OC}$ and $\overrightarrow{OD}$, and the vector $\overrightarrow{MN}$ in terms of the vectors $\overrightarrow{OM}$ and $\overrightarrow{ON}$, where O is the point of intersection of the straight lines AC and BD. Further, the vectors $\overrightarrow{OC}$ and $\overrightarrow{ON}$ can be expressed in terms of the vector $\overrightarrow{AO}$, while the vectors $\overrightarrow{OD}$ and $\overrightarrow{OM}$ in terms of the vector $\overrightarrow{BO}$. Suppose that

$$AO{:}OC = p{:}q \text{ and } BO{:}OD = m{:}n. \tag{1}$$

Then we can express the vector $\overrightarrow{DC}$ in terms of $\overrightarrow{AO}$ and $\overrightarrow{BO}$: $\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = \frac{q}{p}\overrightarrow{AO} - \frac{n}{m}\overrightarrow{BO} = \frac{1}{mp}(mq\overrightarrow{AO} - np\overrightarrow{BO})$.

On the other hand, from the parallelism of the line segments BE and AD it follows that

$$AO{:}ON = DO{:}OB = n{:}m. \tag{2}$$

Then, from the figure and Equalities (2) it follows that $\overrightarrow{ON} = \frac{m}{n}\overrightarrow{AO}$. Analogously, from the parallelism of the line segments AM and BC it follows that $BO{:}OM = CO{:}AO = q{:}p$ and $\overrightarrow{OM} = \frac{p}{q}\overrightarrow{BO}$.

Then we can express the vector $\overrightarrow{MN}$ in terms of $\overrightarrow{AO}$ and $\overrightarrow{BO}$: $\overrightarrow{MN}=\overrightarrow{ON}-\overrightarrow{OM}=\frac{p}{q}\overrightarrow{BO}+\frac{m}{n}\overrightarrow{AO}=\frac{1}{nq}(-np\overrightarrow{BO}+mq\overrightarrow{AO})$. Whence $\overrightarrow{DC}=\frac{nq}{mp}\overrightarrow{MN}$, which geometrically just means that the line segments MN and DC are parallel.

The *second type* comprises problems involving the proof that a given point divides a line segment in a definite ratio (in particular, serves as its midpoint).

In order to prove that the point C divides the line segment AB in a certain ratio $AC:CB = m:n$, it is sufficient: (a) to prove the equality $\overrightarrow{AC}=\frac{m}{n}\overrightarrow{CB}$ or (b) to prove the equality $\overrightarrow{QC}=\frac{n}{n+m}\overrightarrow{QA}+\frac{m}{n+m}\overrightarrow{QB}$, where Q is an arbitrary point.

The proof of the sufficiency of the condition of Item (b) is rather easy. Let $\overrightarrow{QC}=\frac{n}{m+n}\overrightarrow{QA}+\frac{m}{m+n}\overrightarrow{QB}$, then $\left(\frac{1}{m}+\frac{1}{n}\right)\overrightarrow{QC}=\frac{1}{m}\overrightarrow{QA}+\frac{1}{n}\overrightarrow{QB}$. $\frac{1}{m}(\overrightarrow{QC}-\overrightarrow{QA})=\frac{1}{n}(\overrightarrow{QB}-\overrightarrow{QC})$, $\frac{1}{m}\overrightarrow{AC}=\frac{1}{n}\overrightarrow{CB}$, and this just means that $AC:CB = m:n$.

Note also that, carrying out the proof in the reverse order, we can make sure that the condition (b) is necessary for dividing the line segment AB by the point C in the ratio $m:n$.

Let us solve several problems of the second type.

Example 3. In an arbitrary quadrilateral, the line segment joining the midpoints of the diagonals passes through the point of intersection of the midlines. Prove that this segment is bisected by this point.

Solution. The fact that the point O (Fig. 85) is the midpoint of the line segment EF can be proved by various methods.

For instance:

(1) prove that $\overrightarrow{EP}=\overrightarrow{QF}$, which means that $EPFQ$ is a parallelogram, and since EF is its diagonal, it passes through the point O and is bisected by this point;

(2) prove that $\overrightarrow{EO}=\overrightarrow{OF}$;

(3) prove that $\overrightarrow{QO}=\frac{1}{2}(\overrightarrow{QE}+\overrightarrow{QF})$ or $\overrightarrow{NO}=\frac{1}{2}(\overrightarrow{NE}+\overrightarrow{NF})$;

(4) prove that $\overrightarrow{CO}=\frac{1}{2}(\overrightarrow{CE}+\overrightarrow{CF})$ or $\overrightarrow{DO}=\frac{1}{2}(\overrightarrow{DE}+\overrightarrow{DF})$.

Consider the first method of proof which is the simplest in this case.

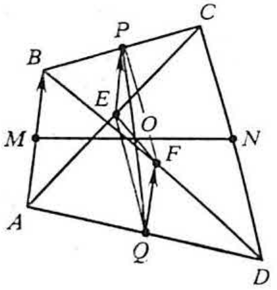


Fig. 85

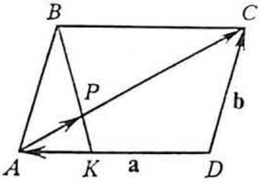


Fig. 86

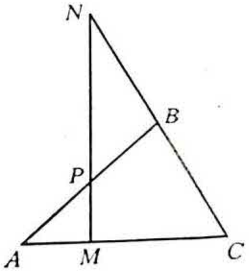


Fig. 87

In the triangle ABC, the line segment EP is a midline, whence $\overrightarrow{EP}=\frac{1}{2}\overrightarrow{AB}$. In the triangle ABD, the line segment QF is a midline, whence $\overrightarrow{QF}=\frac{1}{2}\overrightarrow{AB}$. This means that $\overrightarrow{EP}=\overrightarrow{QF}$. The problem has been proved.

Example 4. The side AD of the parallelogram $ABCD$ is divided into n equal parts and the first division point (point K) is joined to the vertex B (Fig. 86). Find the parts into which the diagonal AC is divided by the half-line BK.

Solution. Let $\overrightarrow{DC}=\mathbf{b}$, $\overrightarrow{DA}=\mathbf{a}$, and $\overrightarrow{AP}=\alpha\overrightarrow{AC}$. We express the vector $\overrightarrow{AP}$ in terms of the vectors $\mathbf{a}$ and $\mathbf{b}$ in two ways: (1) $\overrightarrow{AP}=\alpha\overrightarrow{AC}=\alpha(\mathbf{b}-\mathbf{a})=\alpha\mathbf{b}-\alpha\mathbf{a}$; (2) $\overrightarrow{AP}=\overrightarrow{AK}+\overrightarrow{KP}=-\frac{1}{n}\mathbf{a}+\alpha\overrightarrow{KB}=-\frac{1}{n}\mathbf{a}+\alpha\left(\frac{1}{n}\mathbf{a}+\mathbf{b}\right)=\frac{\alpha-1}{n}\mathbf{a}+\alpha\mathbf{b}$ ($\overrightarrow{KP}=\alpha\overrightarrow{KB}$, since $\triangle APK\sim\triangle BPC$).

Since only one representation of a vector in terms of two noncollinear vectors is possible, we have: $\frac{\alpha-1}{n}=-\alpha$, whence $\alpha=\frac{1}{n+1}$. This means that $\overrightarrow{AP}=\frac{1}{n+1}\overrightarrow{AC}$, and then, as it is easy to see, $AP{:}PC=1{:}n$.

Example 5. Taken on the side AC of the triangle ABC is a point M such that $AM=\frac{1}{3}AC$, and on the extension of the side BC a point N such that $BN=CB$. Find the ratio in which the point P of intersection of the line segments AB and MN divides each of these segments (Fig. 87).

Solution. Let us assume that

$$NP{:}PM=\alpha{:}\beta,\quad AP{:}PB=\gamma{:}\delta. \tag{1}$$

Thus, we have to find $\alpha:\beta$ and $\gamma:\delta$. For this purpose, let us set up several equations containing α, β, γ, and δ.

We may write two such equations using the theorem that a line segment is divided in a given ratio.

Let Q be an arbitrary point in the plane; then for the line segments AB and MN we have:

$$\overrightarrow{QP}=\frac{\beta}{\alpha+\beta}\overrightarrow{QN}+\frac{\alpha}{\alpha+\beta}\overrightarrow{QM}, \tag{2}$$

$$\overrightarrow{QP}=\frac{\delta}{\gamma+\delta}\overrightarrow{QA}+\frac{\gamma}{\gamma+\delta}\overrightarrow{QB}. \tag{3}$$

Equalities (2) and (3) contain five distinct vectors. We reduce their number by replacing them by other vectors and use once again the theorem that a line segment is divided in a given ratio. For the line segments NC and AC we then have: $\overrightarrow{QB}=\frac{1}{2}(\overrightarrow{QN}+\overrightarrow{QC})$,

$$\overrightarrow{QM}=\frac{2}{3}\overrightarrow{QA}+\frac{1}{3}\overrightarrow{QC}. \tag{4}$$

Substituting the values of $\overrightarrow{QB}$ and $\overrightarrow{QM}$ obtained from Equality (4) into Equalities (2) and (3), we get:

$$\overrightarrow{QP}=\frac{\delta}{\gamma+\delta}\overrightarrow{QA}+\frac{\gamma}{2(\gamma+\delta)}\overrightarrow{QN}+\frac{\gamma}{2(\gamma-\delta)}\overrightarrow{QC}, \tag{5}$$

$$\overrightarrow{QP}=\frac{2\alpha}{3(\alpha+\beta)}\overrightarrow{QA}+\frac{\beta}{\alpha+\beta}\overrightarrow{QN}+\frac{\alpha}{3(\alpha+\beta)}\overrightarrow{QC}. \tag{6}$$

Whence we have:

$$\begin{cases}\dfrac{\gamma}{2(\gamma+\beta)}=\dfrac{\beta}{\alpha+\beta}, \\ \dfrac{\delta}{\gamma+\delta}=\dfrac{2\alpha}{3\alpha+\beta}, \\ \dfrac{\gamma}{2(\gamma+\delta)}=\dfrac{\alpha}{3(\alpha+\beta)}.\end{cases}$$

Solving this system of equations, we get: $\gamma=\delta$ and $\beta=\frac{1}{3}\alpha$. Thus, $AP=BP$ and $NP:PM=3:1$.

Consider problems of the *third type*. This type comprises such problems in which it is required to prove that three points belong to one straight line. These problems could be regarded as a particular case of the second type. However, their solution has certain specific properties in connection with the use of the condition that three points belong to one straight line.

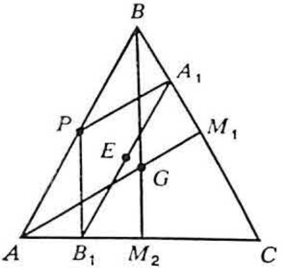


Fig. 88

Example 6. Given on the side AB of the triangle ABC is a point P through which straight lines are drawn parallel to its medians AM_1 and BM_2 and intersecting the sides of the triangle at points A_1 and B_1. Prove that the point E is the midpoint of the line segment A_1B_1, and that the point P and the point G of intersection of the medians of the given triangle lie on one straight line (Fig. 88).

Solution. Let us change the concluding part of the problem so that vectors are applicable to its solution. This can be done in the following ways:

(1) ascertain that $\overrightarrow{EP} = k\overrightarrow{GP}$ (other vectors may also be taken);

(2) for a certain point Q establish that $\overrightarrow{QE} = k\overrightarrow{QP} + (1-k)\overrightarrow{QC}$ (the condition that three points belong to one straight line).

The first way of solution is already known from the above examples of solving problems of the first type.

Consider the second way. To this end, we first deduce the condition that three points belong to one line.

For the points A, B, and C to belong to one straight line, it is necessary and sufficient that for a certain point Q the equality $\overrightarrow{QC} = p\overrightarrow{QA} + q\overrightarrow{QB}$ be fulfilled, where $p + q = 1$.

Proof. Let the points A, B, and C belong to one line. Then we may write that $AC:CB = m:n$. This means the validity of the following equalities: $AC:CB = m:n$, whence $\overrightarrow{QC} = \frac{m}{m+n}\overrightarrow{QA} + \frac{n}{m+n}\overrightarrow{QB}$, $\overrightarrow{QC} = p\overrightarrow{QA} + q\overrightarrow{QB}$, $p + q = 1$.

The above considerations prove both the necessity and sufficiency of the condition.

The solution of the present problem is thus reduced to finding an expression relating the vectors $\overrightarrow{QP}$, $\overrightarrow{QE}$, and $\overrightarrow{QG}$. If the point Q is chosen arbitrarily, then the solution of the problem will turn out to be very complicated. It is best to take it as coinciding with the point C. In this case, the vectors $\overrightarrow{CP}$, $\overrightarrow{CE}$, and $\overrightarrow{CG}$ are readily expressed in terms of $\overrightarrow{CA}$ and $\overrightarrow{CB}$. Indeed, let

$$AP:PB = m:n. \tag{1}$$

Then

$$AB_1:B_1C = m:(m+n+n) = m:(2n+m) \tag{2}$$

(since M_2 is the midpoint of AC) and

$$BA_1:A_1C = n:(m+m+n) = n:(2m+n) \tag{3}$$

(since M_1 is the midpoint of BC). From the property of the centre of gravity G it follows that $\overrightarrow{CG}=\frac{2}{3}\cdot\frac{1}{2}(\overrightarrow{CA}+\overrightarrow{CB})=\frac{1}{3}(\overrightarrow{CA}+\overrightarrow{CB})$. From (2) and (3) we may write that $\overrightarrow{CB_1}=\frac{2n+m}{2(m+n)}\overrightarrow{CA_1}$, and $\overrightarrow{CA_1}=\frac{2m+n}{2(m+n)}\overrightarrow{CB_1}$. Then from the property of the midpoint E of the line segment A_1B_1 we may write that $\overrightarrow{CE}=\frac{1}{2}(\overrightarrow{CB_1}+\overrightarrow{CA_1})=\frac{1}{4}\left(\frac{2n+m}{m+n}\overrightarrow{CA}+\frac{n+2m}{m+n}\overrightarrow{CB}\right)$.

By the theorem on dividing a line segment in a given ratio, we have: $\overrightarrow{CP}=\frac{n}{m+n}\overrightarrow{CA}+\frac{m}{m+n}\overrightarrow{CB}$.

To relate the vectors $\overrightarrow{CG}$, $\overrightarrow{CE}$, and $\overrightarrow{CP}$, let us transform the vector $\overrightarrow{CE}$: $\overrightarrow{CE}=\frac{1}{4}\left(\frac{m+2n}{m+n}\overrightarrow{CA}+\frac{2m+n}{m+n}\overrightarrow{CB}\right)=\frac{1}{4}\left(\overrightarrow{CA}+\overrightarrow{CB}+\frac{n}{m+n}\overrightarrow{CA}+\frac{m}{m+n}\overrightarrow{CB}\right)=\frac{1}{4}(3\overrightarrow{CG}+\overrightarrow{CP})=\frac{3}{4}\overrightarrow{CG}+\frac{1}{4}\overrightarrow{CP}$, that is, $\overrightarrow{CE}=\frac{3}{4}\overrightarrow{CG}+\frac{1}{4}\overrightarrow{CP}$, and since $\frac{3}{4}+\frac{1}{4}=1$, the points E, G, and P belong to one and the same line, and $EG:PE=1:3$. The problem has been solved.

The considered types of affine problems in the plane by far do not exhaust the entire variety of such problems. But they form most numerous groups of problems which justifies their special consideration.

II. Metric Problems

When solving metric problems, we use the scalar product of vectors. Without classifying these problems by types, let us consider several examples.

Example 7. A point P is given on the base AB of the isosceles triangle ABC. (a) Prove that $PC^2 = AC^2 - AP\cdot BP$. (b) Find out how this formula will change if the point P is situated on the extension of the base AB (Fig. 89).

Solution. (a) Let us write the given equality in vector form. Taking into account the directions of the vectors $\overrightarrow{AP}$ and $\overrightarrow{PB}$, we get: $\overrightarrow{PC}^2=\overrightarrow{AC}^2-\overrightarrow{AP}\cdot\overrightarrow{BP}$. Let us prove this equality. We transform its right-hand side in the following way: $\overrightarrow{AC}^2-\overrightarrow{AP}\cdot\overrightarrow{BP}=\overrightarrow{AC}^2-(\overrightarrow{AC}+\overrightarrow{CP})\cdot(\overrightarrow{PC}+\overrightarrow{CB})=\overrightarrow{AC}^2-\overrightarrow{AC}\cdot\overrightarrow{PC}-\overrightarrow{AC}\cdot\overrightarrow{CB}+\overrightarrow{CP}^2-\overrightarrow{CP}\times$

$\overrightarrow{CB} = (\overrightarrow{AC}^2 - \overrightarrow{AC}\cdot\overrightarrow{PC}) - (\overrightarrow{AC}\cdot\overrightarrow{CB} + \overrightarrow{CP}\cdot\overrightarrow{CB}) + \overrightarrow{CP}^2 = \overrightarrow{AC}\,(\overrightarrow{AC} - \overrightarrow{PC}) - \overrightarrow{CB}\cdot(\overrightarrow{AC} + \overrightarrow{CP}) + \overrightarrow{CP}^2 = (\overrightarrow{AC} + \overrightarrow{CP})\cdot(\overrightarrow{AC} - \overrightarrow{CB}) + \overrightarrow{CP}^2 = \overrightarrow{AP}\,(\overrightarrow{AC} - \overrightarrow{CB}) + \overrightarrow{CP}^2$.

If now the vector $\overrightarrow{CB'} = \overrightarrow{AC}$, then $\overrightarrow{AC} - \overrightarrow{CB} = \overrightarrow{CB'} - \overrightarrow{CB} = \overrightarrow{BB'}$, $\triangle AB'B$ is a right triangle, and, thus, $\overrightarrow{AP}\,(\overrightarrow{AC} - \overrightarrow{CB}) = \overrightarrow{AP}\cdot\overrightarrow{BB'} = 0$. Consequently, $\overrightarrow{AC}^2 - \overrightarrow{AP}\cdot\overrightarrow{PB} = \overrightarrow{CP}^2$.

(b) If the point P belongs to the line segment AB, then, when passing from a vector equality to a scalar equality, we have: $\overrightarrow{PC}^2 = |\overrightarrow{PC}|^2 = PC^2$, $\overrightarrow{AC}^2 = |\overrightarrow{AC}|^2 = AC^2$, $\overrightarrow{AP}\cdot\overrightarrow{PB} = |\overrightarrow{AP}|\cdot|\overrightarrow{PB}|\cdot\cos\angle(\overrightarrow{AP}, \overrightarrow{PB}) = AP\cdot PB\cdot\cos 0^\circ = AP\cdot PB$, that is, $PC^2 = AC^2 - AP\cdot PB$.

And if the point P does not belong to the line segment AB, then the vectors $\overrightarrow{AP}$ and $\overrightarrow{PB}$ are in opposite directions, and $\overrightarrow{AP}\cdot\overrightarrow{PB} = |\overrightarrow{AP}|\cdot|\overrightarrow{PB}|\cdot\cos 180^\circ = -AP\cdot PB$. Thus, in this case, the desired equality has the form: $PC^2 = AC^2 + AP\cdot PB$.

Example 8. Find the sum of the squares of the medians of a triangle given its sides a, b, and c.

Solution. Let in the triangle ABC, $\overrightarrow{AB} = \mathbf{c}$, $\overrightarrow{BC} = \mathbf{a}$, and $\overrightarrow{CA} = \mathbf{b}$ (Fig. 90). Then, by the definition of the sum of vectors, $\overrightarrow{AD} = \mathbf{c} + \frac{\mathbf{a}}{2}$, $\overrightarrow{BE} = \mathbf{a} + \frac{\mathbf{b}}{2}$, and $\overrightarrow{CF} = \mathbf{b} + \frac{\mathbf{c}}{2}$.

Using the property of a scalar square, we get: $\overrightarrow{AD}^2 + \overrightarrow{BE}^2 + \overrightarrow{CF}^2 = \left(\mathbf{c} + \frac{\mathbf{a}}{2}\right)^2 + \left(\mathbf{a} + \frac{\mathbf{b}}{2}\right)^2 + \left(\mathbf{b} + \frac{\mathbf{c}}{2}\right)^2 = \mathbf{c}^2 + \mathbf{c}\cdot\mathbf{a} + \frac{\mathbf{a}^2}{4} + \mathbf{a}\cdot\mathbf{b} + \frac{\mathbf{b}^2}{4} + \mathbf{b}^2 + \mathbf{b}\cdot\mathbf{c} + \frac{\mathbf{c}^2}{4} = \frac{5}{4}(a^2 + b^2 + c^2) + (\mathbf{c}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c})$.

Since $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, we have: $(\mathbf{a} + \mathbf{b} + \mathbf{c})^2 = 0$. Thus, $\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + 2\,(\mathbf{c}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c}) = 0$, i.e. $a^2 + b^2 + c^2 = -2\,(\mathbf{c}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c})$.

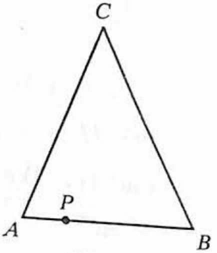


Fig. 89

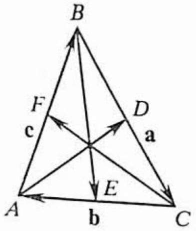


Fig. 90

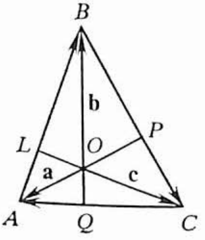


Fig. 91

Hence, $\mathbf{c}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}=-\frac{a^2+b^2+c^2}{2}$.

Substituting the obtained value of the expression $\mathbf{c}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}$, we get: $AD^2+BE^2+CF^2=\frac{3}{4}(a^2+b^2+c^2)$, since, according to the property of a scalar square, $\overrightarrow{AD}^2=AD^2$, $\overrightarrow{BE}^2=BE^2$, and $\overrightarrow{CF}^2=CF^2$.

Example 9. Prove that the altitudes of an arbitrary triangle are concurrent (that is, intersect at a common point).

Solution. Let AP, BQ, and CL be the altitudes of the triangle ABC, and O the point of intersection of the altitudes AP and BQ (Fig. 91).

Let us set for brevity: $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, and $\overrightarrow{OC}=\mathbf{c}$.

By the definition of the difference between vectors, $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$, $\overrightarrow{BC}=\mathbf{c}-\mathbf{b}$, and $\overrightarrow{CA}=\mathbf{a}-\mathbf{c}$. Since, further, $\overrightarrow{OA}\perp\overrightarrow{BC}$, we have: $\mathbf{a}(\mathbf{c}-\mathbf{b})=0$, that is,

$$\mathbf{a}\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{b}. \tag{1}$$

Similarly, since $\overrightarrow{OB}\perp\overrightarrow{CA}$, we have: $\mathbf{b}(\mathbf{a}-\mathbf{c})=0$, whence

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}. \tag{2}$$

By transitivity, it follows from Equalities (1) and (2) that $\mathbf{a}\cdot\mathbf{c}=\mathbf{b}\cdot\mathbf{c}$, or $\mathbf{c}(\mathbf{a}-\mathbf{b})=0$. This means that $\overrightarrow{OC}\perp\overrightarrow{AB}$, or $\overrightarrow{CO}\perp\overrightarrow{AB}$. Since a unique straight line perpendicular to the given line passes through the given point, it follows that CL coincides with CO from the fact that $CO\perp AB$ and $CL\perp AB$. Thus, the three altitudes of the triangle intersect at a common point.

PROBLEMS TO BE SOLVED WITHOUT ASSISTANCE

I. Addition and Subtraction of Vectors. Multiplication of a Vector by a Number

398. For a quadrilateral $ABCD$ to be a parallelogram, it is necessary and sufficient that for any point Q the equality $\vec{QA}+\vec{QC}=\vec{QB}+\vec{QD}$ be fulfilled. Prove this.

399. In a quadrilateral $ABCD$, M, N, P, and Q are the respective midpoints of the consecutive sides. Prove that the quadrilateral $MNPQ$ is a parallelogram.

400. Constructed on the side AB of a quadrilateral $ABCD$ is a parallelogram $ABCC'$, and a point O is taken which is the midpoint of the line segment $C'D$. Prove that if M and N are the respective midpoints of the sides AB and CD, then the line segment AO is equal and parallel to the line segment MN.

401. In a quadrilateral $ABCD$, M and N are the respective midpoints of the sides AD and BC. Prove that $2MN\leqslant AB+CD$.

402. Prove that the length of the line segment connecting the midpoints of the diagonals of a trapezoid is equal to half the difference between its bases.

403. If the length of the line segment joining the midpoints of two opposite sides of a convex quadrilateral is equal to half the sum of two other sides, then this quadrilateral is a trapezoid or a parallelogram. Prove this.

404. Points M and N are the respective midpoints of the sides AB and CD of a quadrilateral $ABCD$. Prove that the midpoints of the diagonals of quadrilaterals $AMND$ and $BMNC$ are the vertices of a parallelogram (or lie on the same straight line).

405. For the point Q to be the centre of gravity of a triangle ABC, it is necessary and sufficient that $\overrightarrow{QA} + \overrightarrow{QB} + \overrightarrow{QC} = \mathbf{0}$. Prove this.

406. Drawn from the point M, lying inside a triangle ABC, are perpendiculars to the sides BC, AC, AB, and laid off on these perpendiculars are the line segments MA_1, MB_1, and MC_1 equal to the respective sides of the triangle. Prove that the point M is the centre of gravity of the triangle $A_1B_1C_1$.

407. Prove that in an arbitrary quadrilateral, the line segment joining the midpoints of the diagonals passes through the point of intersection of the midlines and is bisected by this point.

408. Given a triangle ABC and an arbitrary point Q. Prove that if the parallelograms QBB_1C and QAA_1B_1 are constructed, then the diagonal QA_1 of the parallelogram QAA_1B_1 passes through the centre of gravity O of the given triangle, and $QA_1 = 3QO$.

409. Prove that the straight line passing through the vertex A of a triangle ABC and the midpoint of the median BD divides the side BC in the ratio 1:2.

410. Given two equal line segments AB and A_1B_1. What must be the size of the angle between the lines containing these segments so that the distance between the midpoints of the line segments AA_1 and BB_1 is equal to $\frac{1}{2}AB$?

411. Through the point M taken inside a parallelogram, straight lines are drawn parallel to its sides. They intersect the sides of the parallelogram at points A, C and B, D. Prove that the point of intersection of the midlines of the quadrilateral $ABCD$ is the midpoint of the line segment OM, where O is the centre of the given parallelogram.

412. Prove that the straight line joining the midpoints of the bases of a trapezoid passes through the point of intersection of the extensions of the lateral sides and through the point of intersection of the diagonals.

413. Given a trapezoid $ABCD$. A straight line parallel to its bases AB and CD intersects the lateral sides AD and BC at points M and N, respectively. Prove that if $AN \parallel CM$, then $DN \parallel BM$.

414. Given a trapezoid $ABCD$ in which AB and CD are bases, and M and N are the midpoints of its lateral sides AD and BC. Prove that the straight line AN is not parallel to the straight line CM.

415. Drawn in a circle with centre O are two mutually perpendicular chords AB and CD intersecting at the point M. Prove that the midpoints of the chords AC and BD, the point M, and the centre of the given circle are the vertices of a parallelogram.

416. A median CC_1 is drawn in a triangle ABC. Prove that $CC_1 < \frac{1}{2}(CA + CB)$.

417. Given a triangle ABC. Prove that $OM < \frac{1}{3}(OA + OB + OC)$, where M is the point of intersection of the medians of the triangle, and O an arbitrary point of the plane.

418. Two parallelograms $ABCD$ and $AB_1C_1D_1$ have a common vertex A. Prove that $CC_1 \leqslant BB_1 + DD_1$.

419. Given two parallelograms $ABCD$ and $A_1B_1C_1D_1$. Prove that in the general case, the midpoints of the line segments AA_1, BB_1, CC_1, and DD_1 are the vertices of the parallelogram $A_0B_0C_0D_0$. Construct two parallelograms such that the points A_0, B_0, C_0, and D_0 either coincide or belong to one line.

420. Given a parallelogram $ABCD$. The points P, Q, R, and S divide the sides AB, BC, CD, and DA in equal ratios. Prove that the quadrilateral $PQRS$ is a parallelogram.

421. Given two triangles ABC and $A_1B_1C_1$. Prove that if the medians of the first triangle are parallel to the sides of the second, then the medians of the second triangle are parallel to the sides of the first.

422. Given a quadrilateral $ABCD$. A second quadrilateral is constructed with the vertices at the points of intersection of the medians of the triangles BCD, CDA, DAB, and ABC. Prove that the midlines of the quadrilaterals intersect at a common point.

423. Given in the plane are four straight lines, of which no three lines pass through a common point and no two lines are parallel. Prove that if one of the four given lines is parallel to a median of the triangle determined by the three other lines, then each of the three remaining lines possesses similar properties.

424. Drawn through the vertices A, B, and C of a triangle ABC are the respective straight lines l, m, and n intersecting at the point S. Prove that the straight lines l_1, m_1, and n_1 passing respectively through the midpoints A_0, B_0, and C_0 of the sides BC, CA, and AB parallel to the lines l, m, and n also intersect at a common point.

425. Given a triangle ABC and a point M. Points A_1, B_1, and C_1 are the midpoints of its sides BC, CA, and AB, respectively. Drawn through the points A, B, and C are straight lines parallel to the straight lines MA_1, MB_1, and AC_1, respectively. Prove that these lines are concurrent.

426. Prove that if the length of the midline MN of a quadrilateral $ABCD$ is equal to half the sum of the lengths of its sides AB and CD (M belongs to BC, and N belongs to DA), then $ABCD$ is a trapezoid or a parallelogram.

427. Constructed externally on the sides of a triangle ABC are triangles ABC_1, BA_1C, and CAB_1. Prove that the points of intersection of the medians of the triangles ABC and $A_1B_1C_1$ coincide.

428. Laid off on the altitudes AA_1 and BB_1 of a triangle ABC extended beyond the vertices A and B are line segments AA_2 and BB_2; $AA_2 = BC$ and $BB_1 = AC$. Prove that $CA_2 = CB_2$ and $CA_2 \perp CB_2$.

429. Constructed externally on the sides CA and CB of a triangle ABC are squares CAA_1C_1 and CBB_1C_1'. Prove that the median of the triangle CC_1C_1' drawn through the vertex C is perpendicular to the side AB and is equal to its half.

430. Constructed externally on the sides of a quadrilateral $ABCD$ are squares ABB_1A_1, BCC_1B_1', CDD_1C', and $DAA_1'D_1'$ with centres P, Q, R, and S, respectively. Prove that the line segments PR and QS are equal and mutually perpendicular.

431. Given a quadrilateral $ABCD$. Its midlines intersect at the point M. A broken line $MAUV$ is constructed, where $\overrightarrow{AU} = \overrightarrow{MB}$ and $\overrightarrow{UV} = \overrightarrow{MC}$. Prove that M is the midpoint of the line segment VD. Find the ratio of the area of the quadrilateral $ABCD$ to the area of the quadrilateral $MAUV$.

432. Prove that the medians of a triangle, intersecting at the median point, are divided by this point in the ratio 2:1.

433. Taken on the side AD and on the diagonal AC of a parallelogram $ABCD$ are respective points M and N such that $AM = \frac{1}{5}AD$ and $AN = \frac{1}{6}AC$. Prove that the points B, N, and M lie in one straight line. In what ratio does the point N divide the line segment MB?

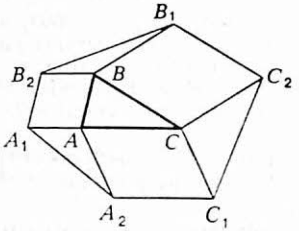


Fig. 92

434. Prove that in an arbitrary quadrilateral $ABCD$, the line segments whose end points are the midpoints of the opposite sides (P, K, R, and L are the midpoints of the sides AB, BC, CD, and AD, respectively) and the line segment whose end points are the midpoints of the diagonals (M is the midpoint of BD) intersect at a common point and are bisected by this point.

435. Given parallel straight lines l_1 and l_2 and two pairs of points, i.e. A_1, A_2 and B_1, B_2. On the given lines find respective points C_1 and C_2 such that $A_1C_1 \parallel A_2C_2$, and $B_1C_1 \parallel B_2C_2$.

436. Constructed on the sides of a triangle ABC are parallelograms ABB_2A_1, BCC_2B_1, and ACC_1A_2 (Fig. 92). Is it possible to construct a triangle whose sides are equal to the line segments B_1B_2, C_1C_2, and A_1A_2?

II. Scalar Product of Vectors

437. Given two sides $AB = a$ and $CD = p$ of a quadrilateral $ABCD$ and the angle α between these sides. Find the length of the line segment joining the midpoints of two other sides of the quadrilateral.

438. In the triangle ABC with sides $AB = 5$, $BC = 2$, and $AC = 4$ compute the angle ABC.

439. Prove that the altitudes of an obtuse triangle intersect.

440. Prove that in a parallelogram, the sum of the squares of its diagonals is equal to the sum of the squares of its sides.

441. Prove that if $ABCD$ is a rectangle, then for any point M the equality $MA^2 + MC^2 = MB^2 + MD^2$ holds true.

442. Prove that the angle C of a triangle ABC will be acute, right or obtuse according as the median CD is greater than, equal to or less than $\frac{1}{2}AB$.

443. Prove that the relationship $AB^2 + BC^2 + AC^2 = 3(MA^2 + MB^2 + MC^2)$ holds true in the triangle ABC with centre of gravity M.

444. Prove that if the centre of gravity of a triangle ABC coincides with the point of intersection of the altitudes, then the triangle is equilateral.

445. Express each median of a triangle in terms of its sides.

446. In a trapezoid with mutually perpendicular diagonals, the larger base is equal to 4, and the smaller to 3. Find its lateral side if it is known that it forms an angle of 60° with the larger base.

447. Given a triangle ABC in which $AC = 4$, $BC = 3$, and $\angle ABC = 120°$. Find the distance from the vertex C to the point M dividing the side AB in the ratio 1:3 as measured from the vertex A.

448. Prove that if the altitude CD of a right triangle ABC is drawn from the vertex of the right angle, then: (a) $CD^2 = AD \cdot BD$; (b) $AC^2 = AB \cdot AD$; (c) $BC^2 = BA \cdot BD$.

449. The diagonals of a right trapezoid are mutually perpendicular. Prove that the altitude of the trapezoid is a mean proportional between its bases.

450. Prove that in the trapezoid $ABCD$ with bases AB and CD, the following equality is fulfilled: $AC^2 + BD^2 > AD^2 + BC^2 + 2AB \cdot DC$.

451. For the diagonals of a quadrilateral to be mutually perpendicular, it is necessary and sufficient that the sum of the squares of the opposite sides of the quadrilateral be equal. Prove this.

452. Prove that if two medians of a triangle are mutually perpendicular, then the sum of their squares is equal to the square of the third median.

453. Find the relationship among the sides of a triangle ABC if its medians AA_1 and BB_1 are mutually perpendicular.

454. The legs of a right triangle are equal to a and b. Find the angle bisector drawn from the vertex of the right angle.

455. A perpendicular DM is drawn to the side BC from the midpoint D of the base AB of an isosceles triangle ABC. Point N is the midpoint of the line segment MD. Prove that the line segments AM and CN are mutually perpendicular.

456. A straight line is drawn through the vertex of the right angle C of a triangle ABC. Perpendiculars AA_1 and BB_1 are dropped from the vertices A and B to this line. The vertex C is reflected in the point C_1 with respect to the midpoint M of the line segment A_1B_1. Prove that $\angle AC_1B = \frac{\pi}{2}$.

457. Constructed on the side AB of a triangle ABC on different sides of the straight line AB are equilateral triangles ABC_1 and ABC_2. Find the relationship among the sides of the given triangle if the lines CC_1 and CC_2 are mutually perpendicular ($C \neq C_1$ and $C \neq C_2$).

III. Miscellaneous Problems

458. Constructed externally on the sides of a parallelogram are squares. Prove that the centres of these squares are the vertices of a square.

459. Constructed externally on the sides AB and BC of a triangle ABC are squares $ABDE$ and $BCKF$. Prove that the line segment DF is twice the length of the median BP of the triangle ABC and perpendicular to it.

460. Constructed externally on the sides AB and BC of a triangle ABC are equilateral triangles ABC_1 and BCA_1. Prove that the line segment joining the midpoints of the line segments AB and A_1B_1 is equal to half the length of the line segment AC and forms an angle of 60° with this segment.

461. Constructed externally on the sides AB and BC of a triangle ABC are equilateral triangles ABC_1 and BCA_1. Prove that if M, N, and P are the respective midpoints of the sides AC, C_1B, and BA_1, then triangle MNP is equilateral.

462. The diagonals AC and BD of an isosceles trapezoid $ABCD$ ($AB \parallel CD$) intersect at the point O at an angle of 60°. Prove that the midpoints of the line segments OA, OD, and BC are the vertices of an equilateral triangle.

463. The diagonal AC of a trapezoid $ABCD$ cuts off an equilateral triangle ACD. From the point E of the diagonal AC (or its extension), the base BC is seen at an angle of 60°. Prove that the midpoints of the line segments AE, BC, and CD are the vertices of an equilateral triangle.

464. If regular triangles are constructed (externally or internally) on two sides of a parallelogram emanating from one and the same vertex, then the opposite vertex of the parallelogram and the free vertices of the triangles form a regular triangle. Prove this.

465. Given two equally oriented equilateral triangles $A_1B_1C_1$ and $A_2B_2C_2$. The line segments A_1A_2, B_1B_2, and C_1C_2 are divided by the points A, B, and C in the same ratio from the end points A_1, B_1, and C_1, respectively. Prove that the triangle ABC is equilateral.

466. In a right trapezoid $ABCD$ with an acute angle of 45°, the diagonal AC is equal to the side CD. Prove that the midpoint of the smaller base is equidistant from the vertex A and the midpoint of the side CD.

467. Constructed on the line segments AB and AC of a straight line are right isosceles triangles ABC_1 and ACB_1 ($\angle C_1 = \angle B_1 = 90°$) having opposite orientations. Prove that the midpoint of the line segment BC and the points B_1 and C_1 serve as the vertices of a right isosceles triangle.

468. Drawn in a triangle ABC ($\angle B = 45°$) are the altitudes CC_1 and AA_1 intersecting at the point O. Prove that the midpoints of the line segments BC, A_1C_1, and CD serve as the vertices of a right isosceles triangle.

469. Constructed externally on the sides AB and BC of a triangle ABC are equilateral triangles ABC_1 and BCA_1 with centres O_1 and O_2, respectively. Prove that the line segment O_1O_2 is twice the length of the line segment joining the midpoints of the line segments O_1C and C_1O_2 and makes an angle of 60° with this segment.

470. Constructed externally on the sides AB, BC, and CD of a rectangle $ABCD$ are equilateral triangles ABO_1, BCO_2, and CDO_3. Prove that the distances between the line segments AB, O_1O_2 and BC, O_2O_3 are equal.

471. Squares $ABMN$ and $CDKL$ are constructed externally on the sides AB and CD of an arbitrary convex quadrilateral $ABCD$. Prove that the midpoints of the diagonals of the quadrilaterals $ABCD$ and $MNKL$ are the vertices of a square or coincide.

472. Given two equally oriented squares $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$. The line segments A_1A_2, B_1B_2, C_1C_2, and D_1D_2 are divided by the points A_0, B_0, C_0, and D_0 in the same ratio beginning with the vertices of one of these squares. Prove that the quadrilateral $A_0B_0C_0D_0$ is a square.

473. Given two regular like polygons $A_1A_2 \ldots A_n$ and $B_1B_2 \ldots B_n$ of the same orientation. The line segments A_1A_2, B_1B_2, $\ldots$, A_nB_n are divided by the points C_1, C_2, $\ldots$, C_n, respectively, in the same ratio beginning with the vertices of one of these polygons. Prove that the polygon $C_1C_2 \ldots C_n$ is regular.

474. Constructed externally on the sides AB and BC of a triangle ABC are right isosceles triangles ABD and BCE ($\angle B = \angle C = 90^\circ$) of the same orientation. Prove that the midpoints of the line segments AB, BC, and DE are the vertices of a right isosceles triangle.

475. In a square $ABCD$, O is its centre, M and N the midpoints of the line segments BO and CD, respectively. Prove that AMN is a right isosceles triangle.

476. Constructed externally on the sides of a quadrilateral $ABCD$ are right isosceles triangles ABM, BCN, CDP, and DAQ ($\angle M = \angle N = \angle P = \angle Q = 90^\circ$). Prove that the midpoints of the line segments MP and NQ and the midpoints of the diagonals of the quadrilateral are the vertices of a square.

477. In a right triangle ABC, the altitude CD is drawn from the vertex of the right angle. The points M and N divide the respective sides AC and CB in equal ratios (as measured from the end points A and C). Prove that the triangle DMN is similar to the given triangle.

478. Squares are constructed externally on the base and one of the lateral sides of an isosceles triangle. Prove that the centres of these squares and the midpoint of the other lateral side serve as the vertices of a right isosceles triangle.

479. In a rectangle $ABCD$, a perpendicular BK is dropped to the diagonal AC. The points M and N bisect the line segments AK and CD, respectively. Prove that $\angle BMN = 90^\circ$.

480. Constructed for a right triangle ABC is a triangle ABC_1 symmetric to the triangle ABC with respect to the hypotenuse AB. If M is the midpoint of the altitude C_1D of the triangle ABC, and N is the midpoint of the side BC, then the triangle AMN is similar to the triangle ABC. Prove this.

481. Given a parallelogram $ABCD$. Chosen on the lines AB and BC are points H and K such that the triangles KAB and HCB are isosceles ($KA = AB$ and $HC = CB$). Prove that the triangle KDH is also isosceles.

482. A quadrilateral $ABCD$ is rotated about a certain point O lying in its plane through an angle of 90° to occupy the position $A_1B_1C_1D_1$. Prove that if P, Q, R, and S are the midpoints of the line segments A_1B, B_1C, C_1D, and D_1A, then the line segments PR and QS are mutually perpendicular and equal.

483. Squares are constructed externally on the sides of a quadrilateral. Prove that the centres of these squares are the vertices of a quadrilateral with equal and mutually perpendicular diagonals.

484. In a right triangle ABC, drawn from the vertex of the right angle is the altitude CD, and a point D_1 is constructed which is symmetric to the point

D with respect to the leg AC. Prove that the point A and the midpoints of the line segments D_1C and CB serve as the vertices of a triangle similar to the given triangle.

485. Constructed in a right triangle ABC is a point D_1 symmetric to some point D of the leg BC with respect to the hypotenuse AB, E is the point of intersection of the line segments DD_1 and AB_1, M and N are the respective midpoints of the line segments AD_1 and CE. Prove that $\angle MNB = 90°$.

486. In a triangle ABC, the altitudes AA_1 and BB_1 are drawn, and a point A_2 is constructed symmetric to the point A_1 with respect to the straight line AC, M and N being the respective midpoints of the line segments B_1A_2 and AB. Prove that CMN is a right triangle.

487. Described on the side AB of a triangle ABC as on its diameter is a circle intersecting the lines AC and BC at points A_1 and B_1, respectively. Prove that the midpoints of the chords AB_1 and BA_1, and the foot of the altitude drawn from the vertex C in the given triangle form a triangle similar to the given one.

488. Perpendiculars MA_1 and MB_1 are dropped from an arbitrary point M taken on the circle circumscribed about a triangle ABC to its sides BC and AC, P and Q being the respective midpoints of the line segments AB and A_1B_1. Prove that $\angle PQM = 90°$.

489. The angle bisector AD is drawn from the vertex A of a triangle ABC to intersect the circle circumscribed about the given triangle at A_1, M and N being the respective midpoints of the line segments CD and A_1B. Prove that the triangles ACA_1 and AMN are similar to each other.

490. A common chord of two intersecting circles is a diameter of one of them. Tangents are drawn through one of the end points of this diameter to the given circles. Prove that the other end point of the diameter and the midpoints of the line segments of the drawn tangents cut off by the circles serve as the vertices of a right triangle.

491. The altitudes AD and BE of a triangle ABC are extended beyond the vertices A and B, and laid off on their extensions are line segments AM and BN such that $AM = BC$ and $BN = AC$. Prove that the line segments CM and CN are mutually perpendicular and equal in length.

492. Constructed externally on the sides AC and BC of a triangle ABC are equilateral triangles ACB_1 and BCA_1, M is the midpoint of the side AB, and O is the centre of the triangle ACB_1. Determine the angles of the triangle MA_1O.

493. Constructed externally on the sides AC and BC of a triangle ABC are squares $ACDA_1$ and $BCEB_1$. Prove that the point of intersection of the lines AB_1 and BA_1 lies on the altitude of the given triangle drawn to the side AB.

494. Given three equilateral triangles A_1BC, A_2DE, and A_3FQ having the same orientation, the points A_1, A_2, and A_3 being the vertices of an equilateral triangle of the same orientation. Prove that the midpoints of the line segments CD, EF, and QB are the vertices of an equilateral triangle.

495. Given a parallelogram $ABCD$. Constructed externally on its sides CD and BC are equally oriented similar triangles CDE and FBC. Prove that the triangle FAE is similar to them and of the same orientation.

496. Constructed on the sides AB, AC, and BC of a triangle ABC as on the bases are three similar isosceles triangles ABP, ACQ, and BCR. The first two are arranged outside the given triangle, while the third one is found on the same side of BC as the given triangle (or conversely). Prove that the quadrilateral $APBQ$ is a parallelogram.

497. Constructed externally on the sides AC and BC of a triangle ABC are similar rectangles $ACMN$ and $BCPQ$. Prove that the point of intersection of the lines NB and QA lies on the altitude of the triangle (or on its extension) drawn from the vertex C.

498. A tangent is drawn at the end point A of the chord AB of the circle O. A perpendicular BM is dropped from the point B to this tangent and meets the

circle for the second time at the point C. Prove that the centre O, the point N dividing the chord AB in the ratio $AN:NB = 1:2$, and the point C' symmetric to the point C with respect to the point M lie in one straight line.

499. The opposite sides AB and CD of a quadrilateral $ABCD$ are divided by the respective points M and N in equal ratios as measured from the points A and C. Prove that the line segment MN divides the midline of the quadrilateral in the same ratio and is bisected itself by the midline.

500. Points P, Q, R, and S divide the sides of a quadrilateral $ABCD$ so that $AP:PB = DQ:QC = m$ and $AR:RD = BS:SC = n$. Prove that the line segments PQ and RS divide each other in the same ratios.

501. A parallelogram $ADEF$ is inscribed in a triangle ABC so that the vertices D, E, and F lie on the sides AB, BC, and AC, respectively. Drawn through the midpoint M of the side BC is a straight line AM intersecting the straight line DE at the point K. Prove that the quadrilateral $CFDK$ is a parallelogram.

502. Straight lines are drawn through the opposite vertices of a parallelogram. These lines intersect its sides or their extensions at four points. Prove that these points are the vertices of a trapezoid or a parallelogram.

503. An arbitrary point M of the lateral side AB of a trapezoid $ABCD$ is joined to the vertices C and D. Drawn from the vertices A and B are straight lines AN and BN parallel to the lines CM and DM, respectively. Prove that the point N of their intersection belongs to the side CD.

504. Given a quadrilateral $ABCD$. A straight line drawn through the vertex A parallel to the side BC intersects the diagonal BD at the point M, and a straight line drawn through the vertex B parallel to the side AD intersects the diagonal AC at the point N. Prove that $MN \parallel CD$.

505. Given an arbitrary central-symmetric hexagon. Constructed externally on its sides as on the bases are regular triangles. Prove that the midpoints of the line segments connecting the vertices of neighbouring triangles are the vertices of a regular hexagon.

506. Taken on the side AC of a triangle ABC is a point M such that $AM = \frac{1}{3}AC$, and taken on the extension of the side BC is a point N such that $BN = CB$. In what ratio does the point of intersection of the line segments AB and MN divide each of these line segments?

507. Given three line segments A_1A_2, B_1B_2, and C_1C_2. Let their midpoints be denoted by A_3, B_3, and C_3, respectively, and the centres of gravity of the triangles $A_1B_1C_1$, $A_2B_2C_2$, and $A_3B_3C_3$ by M_1, M_2, and M_3, respectively. Prove that M_3 is the midpoint of the line segment M_1M_2 (or that the points M_1, M_2, and M_3 coincide).

508. Given on the median CM of a triangle ABC is a point N. Drawn through this point are straight lines AN and BN intersecting the sides BC and AC at points A_1 and B_1, respectively. Prove that the line segment A_1B_1 is bisected by the median CM and is parallel to the side AB.

509. Given on the side AB of a triangle ABC is a point P through which straight lines are drawn parallel to its medians AM_1 and BM_2 intersecting the corresponding sides of the triangle at points A_1 and B_1. Prove that the midpoint of the line segment A_1B_1, the point P_1, and the point Q of intersection of the medians of the given triangle lie in one straight line.

510. The distance from the median point of a triangle to the centre of the circle circumscribed about the triangle is equal to one-third of the radius of this circle. Prove that this triangle is right-angled.

511. Given on two straight lines are line segments AB and CD. The line segment AB is divided by points M and M_1 in the ratios $AM:AB = BM_1:AB = AC:BD$, $AC \neq BD$, and the line segment CD by points N and N_1 in the same ratios, respectively. Prove that the line segment MM_1 is perpendicular to the line segment NN_1.

512. Prove that if the lateral sides of a trapezoid are mutually perpendicular, then the sum of the squares of its bases is equal to the sum of the squares of its diagonals.

513. If in a quadrilateral, the sum of the squares of its diagonals is equal to the sum of the squares of all the sides, then this quadrilateral is a parallelogram. Prove this.

514. Given two equally oriented squares $ABCD$ and $A_1B_1C_1D_1$. Prove that $AA_1^2 + CC_1^2 = BB_1^2 + DD_1^2$.

515. Given on the median CM_3 of a triangle ABC is a point P through which straight lines AP and BP are drawn intersecting the sides CB and AC at points A_1 and B_1, respectively. Prove that if $AA_1 = BB_1$, then the triangle is isosceles.

516. Given on the base AB of an isosceles triangle ABC is a point P. Prove that $PC^2 = AC^2 - AP \cdot BP$. Find out how the formula will change if the point P lies on the extension of the base AB.

517. Prove that if perpendiculars MX, MY, and MZ are dropped from an arbitrary point M taken inside a right triangle ABC (C is a right angle) to the respective sides BC, CA, and AB, then the following relationship holds: $AY \cdot AC + BZ \cdot BA + CX \cdot CB = AB^2$.

518. Taken on the extensions of the sides AB, BC, and CA of a triangle ABC are respective points M, N, and P such that $BM = AB$, $CN = BC$, and $AP = CA$. Compute the ratio of the sum of the squares of the sides of the triangle PMN to the sum of the sides of the triangle ABC.

519. The lateral sides BC and AD of a trapezoid $ABCD$ are rotated about their midpoints through an angle of 90° in the positive direction to occupy the position of the line segments B_1C_1 and A_1D_1. Prove that $D_1C_1 = A_1B_1$.

520. Constructed on the sides AB, CD, and EF of a central-symmetric hexagon are equally oriented equilateral triangles ABP, CDQ, and EFR. Prove that the triangle PRQ is equilateral (in particular, it may degenerate into a point).

521. The side AC of a triangle ABC is rotated about the vertex A through an angle of $+90°$, and the side BC is rotated about the vertex B through an angle of $-90°$. Prove that the position of the midpoint of the line segment C_1C_2 joining the end points C_1 and C_2 of the rotated segments is independent of the position of the vertex C.

522. Constructed on the sides of a quadrilateral as on its diameters are semicircles, two opposite semicircles being arranged internally and two others externally. Prove that the midpoints of these semicircles are the vertices of a parallelogram.

523. Given a square. All possible right isosceles triangles are constructed. The vertex of one of the acute angles of the triangles coincides with a vertex of the square, and the vertex of the right angle belongs to its diagonal. Find the set of the third vertices of the triangles under consideration.

524. Constructed externally on the sides of an arbitrary triangle are squares. Prove that the altitudes of the triangle whose vertices are the centres of these squares pass through the respective vertices of the given triangle.

525. Drawn in a triangle ABC are the altitudes AA_1, BB_1, and CC_1, A_0, B_0, and C_0 being the midpoints of these altitudes. Prove that the triangles $A_0B_0C_1$, $B_0C_0A_1$, and $A_0C_0B_1$ are similar.

526. Given parallel straight lines q_1 and q_2 and two pairs of points, i.e. A_1, A_2 and B_1, B_2. On the given lines find respective points C_1 and C_2 such that $A_1C_1 \parallel A_2C_2$ and $B_1C_1 \parallel B_2C_2$.

527. The sides BC and AD of a quadrilateral $ABCD$ are divided into equal parts by the points B_1, B_2 and A_1, A_2, respectively. Is it always possible to draw a straight line so that its segment enclosed between the sides AB and CD is divided into equal parts by the lines A_1B_1 and A_2B_2?

528. Given three points, A_1, B_1, and C_1. Taking them for division points

of the appropriate sides of some triangle ABC which divide them in the ratio 2:1 in the same direction of traverse, construct the triangle ABC.

529. On the hypotenuse of a right triangle or on its extension, find a point such that the line joining its projections on the legs is perpendicular to the hypotenuse.

530. Let OA, OB, and OC be three rays intersecting two straight lines a and b at points A, B, C and A_1, B_1, C_1, respectively, so that $AB:BC = n$ and $OA:AA_1 = m$. Find the relation between the ratios $OB:OB_1 = x$ and $OC:OC_1 = y$.

531. The opposite sides AB and DC, AD and BC of a quadrilateral $ABCD$ intersect at points E and F, respectively. Prove that the line segments thus formed satisfy the equality $\frac{AE\cdot CE}{BE\cdot DE}=\frac{AF\cdot CF}{BF\cdot DF}$.

532. A straight line passing through the centre of gravity of a triangle divides its sides into some segments. Find the relation between the ratio of the lengths of the segments of one side and the ratio of the lengths of the segments of the other side.

533. Prove that if the extensions of the opposite sides of a quadrilateral intersect pairwise, then the midpoint of the line segment joining these points of intersection lies on one straight line with the midpoints of the diagonals.

534. Prove that the sum of the fourth powers of the distances of a given point situated in the plane of a certain circle to the vertices of any square inscribed in it is constant.

SEC. 7. GREATEST AND LEAST VALUES

Problems on finding greatest and least values are usually successfully solved according to the following scheme:

1. Reveal the quantity to be optimized (that is, the quantity whose greatest or least value is required to be found) and denote it, say, by the letter y (or S, P, r, R, etc. depending on the plot of the problem).
2. One of the unknown quantities (a side, an angle, etc.) is declared to be an independent variable and is denoted by the letter x; real (in accordance with the conditions of the problem) bounds of change of x are set up.
3. Proceeding from the concrete conditions of the problem, express the value of y in terms of x and the known quantities, that is, those given by the hypothesis of the problem (this is the geometrical step of the solution).
4. For the function $y = f(x)$ obtained during the preceding step, find the greatest or least value (depending on the requirements of the problem) within the interval of real change of x found in Item 2.
5. Interpret the result of Item 4 for the given concrete geometrical problem.

During the first three steps, a so-called mathematical, that is, analytical, model of a given geometrical problem is set up. Here, a successful solution depends on a reasonable choice of the independent variable. It is of importance here to express analytically y in terms of x in a comparatively easy way. During the fourth step, the

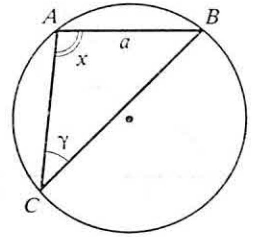


Fig. 93

set-up mathematical model is investigated most frequently by means of mathematical analysis, sometimes elementary techniques being applied. During such an analysis, the geometrical problem itself which served as a starting point for the mathematical model is of no interest to the investigator. And no sooner than the solution of the problem is completed within the frames of the set-up mathematical model, the obtained result is interpreted for the original geometrical problem (the fifth step).

Let us recall the scheme for solving a problem on finding the greatest or the least value of the function $y = f(x)$, differentiable on the interval X, using differential calculus:

(1) find $f'(x)$;

(2) find the stationary and critical points for the function $f(x)$, that is, the points at which $f'(x) = 0$ or $f'(x)$ is not existent, respectively; choose those points which belong to the interval X;

(3) compile a table of the values of the function $y = f(x)$; this table must contain the values of the function at the points found in Item 2 and also at the end points of the interval X. If the interval X does not contain its end points, then the limits of the function $f(x)$ at its end points are entered in the table.

The reader should bear in mind that there are cases when a problem is solved simpler using a pure geometrical method (see Example 5 given below).

Example 1. Given on a circle of radius R are points A and B, the distance between which is equal to a, and an arbitrary point C. Determine the greatest value of the expression $AC^2 + BC^2$ (Fig. 93).

Solution. 1. The expression $AC^2 + BC^2$ is the quantity to be optimized; let us set $AC^2 + BC^2 = y$.

2. Let us choose an independent variable: we set $x = \angle CAB$. The real bounds (limits) of this variable are: $0 < x < \pi - \gamma$, where $\gamma = \angle ACB$ (this angle is independent of the choice of the point C, since it is always measured by half of the minor arc AB); obviously, according to the sense of the problem, the point C must be chosen on the major arc AB.

3. Express y, that is, $AC^2 + BC^2$, in terms of x, a, and R. By the law of sines, $BC = 2R \sin x$ and $AC = 2R \sin(\pi - x - \gamma) = 2R \sin(x + \gamma)$. Since $AB = 2R \sin\gamma$, we get: $a = 2R \sin\gamma$, whence we find: $\sin\gamma = \frac{a}{2R}$. As a result, we get: $y = AC^2 + BC^2 = (2R \sin x)^2 + (2R \sin(x + \gamma))^2 = 4R^2 (\sin^2 x + \sin^2 (x + \gamma))$, where $\sin\gamma = \frac{a}{2R}$ (a mathematical model of the problem has been set up).

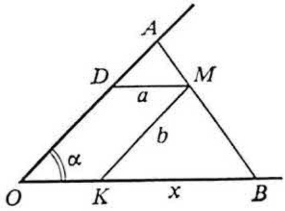


Fig. 94

4. Consider the function $y = 4R^2(\sin^2 x + \sin^2(x+\gamma))$. We have to find its greatest value on the interval $(0, \pi-\gamma)$. Let us make some transformations of the expression representing the function. We have: $y = 4R^2\left(\frac{1-\cos 2x}{2} + \frac{1-\cos(2x+2\gamma)}{2}\right) = 2R^2[2 - (\cos 2x + \cos(2x+2\gamma))] = 4R^2(1-\cos(2x+\gamma)\cos\gamma)$.

The greatest value of the obtained expression can be found without using the derivative: it is clear that it is reached at the point where $\cos(2x+\gamma)$ reaches the least value, that is, when $\cos(2x+\gamma) = -1$. This will happen for $2x+\gamma = \pi$, that is, for $x = \frac{\pi-\gamma}{2}$. Note that the point $\frac{\pi-\gamma}{2}$ belongs to the interval $(0, \pi-\gamma)$.

Let us compute the greatest value of the function y: $y = 4R^2(1-(-1)\cos\gamma) = 4R^2(1+\cos\gamma) = 4R^2(1+\sqrt{1-\sin^2\gamma}) = 4R^2\left(1+\sqrt{1-\frac{a^2}{4R^2}}\right) = 2R(2R+\sqrt{4R^2-a^2})$ (this is the end of the step of solving the problem within the framework of the set-up mathematical model).

5. Returning to the original problem we draw the following conclusion: the greatest value of the expression $AC^2 + BC^2$ is equal to $2R(2R + \sqrt{4R^2 - a^2})$; it is reached when $\angle CAB = \frac{\pi-\gamma}{2}$, that is, when the triangle ABC is isosceles ($AC = CB$).

Example 2. Through a fixed point M inside an angle, draw a straight line cutting off a triangle having the least area (Fig. 94).

Solution. 1. The quantity to be optimized is represented by the area S of the triangle AOB.

2. Draw $DM \parallel OB$ and $MK \parallel OA$. Set $KB = x$; the real bounds within which x changes are: $0 < x < +\infty$.

3. Since M is a fixed point, the line segments DM and KM are also fixed, we set $DM = a$, $KM = b$, and express S in terms of x, a, and b.

Consider the triangles MKB and AOB. They are similar, hence, $\frac{MK}{AO} = \frac{KB}{OB}$, i.e. $\frac{b}{AO} = \frac{x}{a+x}$. Hence we find that $AO = \frac{b(a+x)}{x}$. Further we have: $S = \frac{1}{2}AO \cdot OB \cdot \sin\alpha$, where $\alpha = \angle AOB$. Hence, $S = \frac{1}{2}\frac{b(a+x)}{x}(a+x)\sin\alpha = \frac{b\sin\alpha}{2}\cdot\frac{(a+x)^2}{x}$ (a mathematical model of the problem has been set up).

4. Consider the function $S=k\frac{(a+x)^2}{x}$, $0<x<+\infty$, where $k=\frac{b\sin\alpha}{2}$. Find its least value.

(1) $S'=k\frac{2(a+x)x-(a+x)^2}{x^2}=k\frac{(a+x)(x-a)}{x^2}$.

(2) The derivative does not exist at the point $x=0$ and vanishes at the points $x=-a$ and $x=a$. Of these three points, only the point $x=a$ belongs to the interval $(0, +\infty)$.

(3) We find the one-sided limits of the function at the end points of the interval

$$\lim_{x\to+0}\frac{k(a+x)^2}{x}=+\infty,\quad \lim_{x\to+\infty}\frac{k(a+x)^2}{x}=+\infty.$$

The table of values looks as follows:

x	0	$+\infty$	a
y	$+\infty$	$+\infty$	$4ka$

Hence, the least value of the function is reached at the point $x=a$.

5. Let us return to the original geometrical problem. If $x=KB=a$, then, since $OK=a$ and MK is the midline of the triangle AOB, M is the midpoint of AB. Thus, in order to cut off a triangle having the least possible area from the sides of the angle, we have to draw a straight line through the point M so that its segment enclosed between the sides of the angle is bisected by the point M.

Example 3. Taken on the equal sides AB and BC of an isosceles triangle ABC are points D and E such that $DE \parallel AC$. Constructed on DE as on the base is a square so that the square and point B lie on opposite sides of the line DE. Find the greatest value of the area of the intersection of the triangle and square if $AC=b$, and the altitude BH of the triangle ABC is equal to h.

Solution. 1. The quantity to be optimized is represented by the area S of the intersection of the triangle and square.

2. Let us denote the side of the square by x: $x=DE$ and find the real bounds of change of x. It is clear that of all the squares entirely lying inside the triangle, the greatest possible area belongs to the

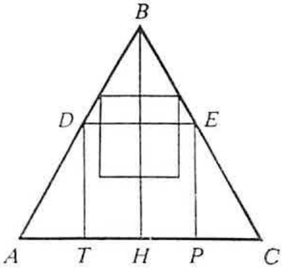


Fig. 95

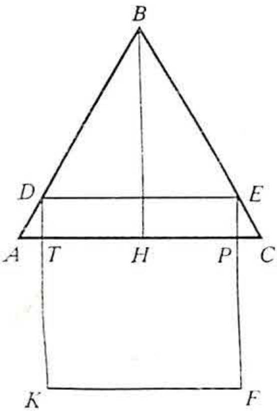


Fig. 96

inscribed square, that is, to the square all the vertices of which lie on the sides of the triangle (Fig. 95). If x exceeds the side of the inscribed square, then the square and triangle are arranged as is shown in Fig. 96. In this case, the intersection of the square and triangle is represented by the inscribed rectangle $DEPT$, the inscribed square being a particular case. Hence, x changes from the side of the inscribed square to the side AC.

Let us find the side of the inscribed square. From the similarity of the triangles BDE and ABC (see Fig. 95) we get: $\frac{x}{b}=\frac{h-x}{h}$, whence we find: $x=\frac{bh}{b+h}$. Thus, $\frac{bh}{b+h}\leqslant x<b$.

3. We express the area S of the inscribed rectangle $DEPT$ in terms of x, a, and h. We get from the similarity of the triangles ADT and ABH: $\frac{DT}{BH}=\frac{AT}{AH}$, that is, $\frac{DT}{h}=\frac{\frac{b}{2}-\frac{x}{2}}{\frac{b}{2}}$, whence $DT=\frac{h(b-x)}{b}$, and consequently, $S=\frac{hx(b-x)}{b}$.

4. Consider the function $S=\frac{h}{b}(bx-x^2)$ on the half-interval $\left[\frac{bh}{b+h}, b\right)$ and find its greatest value:

(1) $S'=\frac{h}{b}(b-2x)$; (2) $S'=0$ for $x=\frac{b}{2}$.

Now, find out whether the point $\frac{b}{2}$ lies inside the half-interval $\left[\frac{bh}{b+h}, b\right)$, that is, whether the inequality $\frac{bh}{b+h}<\frac{b}{2}$ is fulfilled. It is fulfilled if $2h<b+h$, that is, if $h<b$. If $h\geqslant b$, then there are no stationary points inside the half-interval $\left[\frac{bh}{b+h}, b\right)$.

(3) Let us compile a table of values of the function among which we have to look for the greatest one. First of all let us note that

$$\lim_{x\to b-0} S(x)=\lim_{x\to b-0}\frac{h}{b}(bx-x^2)=0.$$

Further, $S\left(\frac{bh}{b+h}\right)=\left(\frac{bh}{b+h}\right)^2$, since $\frac{bh}{b+h}$ is the side of an inscribed square. Finally, $S\left(\frac{b}{2}\right)=\frac{h}{b}\left(b\cdot\frac{b}{2}-\left(\frac{b}{2}\right)^2\right)=\frac{hb}{4}$. If $h<b$, then the table has the following form:

x	$\frac{bh}{b+h}$	$\frac{b}{2}$	b
S	$\left(\frac{bh}{b+h}\right)^2$	$\frac{bh}{4}$	0

Let us prove that $\frac{bh}{4}>\left(\frac{bh}{b+h}\right)^2$. It is reduced to the inequality $(b+h)^2>4bh$, that is, $(b-h)^2>0$, which is an obvious inequality.

Thus, if $h<b$, then the greatest value of the function S is equal to $\frac{bh}{4}$ and is reached at the point $x=\frac{b}{2}$.

If $h\geqslant b$, then the table has the form

x	$\frac{bh}{b+h}$	b
S	$\left(\frac{bh}{b+h}\right)^2$	0

In this case, the greatest value of the function S is equal to $\left(\frac{bh}{b+h}\right)^2$ and is attained at the point $x=\frac{bh}{b+h}$.

5. Returning to the original problem, we arrive at the following conclusion: if the altitude of the triangle is less than the base, then the greatest area will be possessed by the intersection of the triangle and the square constructed on the midline of the triangle. And if the altitude of the triangle is not less than the base, then the area of the square inscribed in the triangle will be the greatest.

Example 4. All possible trapezoids inscribed in a circle of radius R are under consideration. Find the lateral side of the trapezoid having the greatest area if it is known that one of the bases of the trapezoid is equal to $R\sqrt{3}$.

Solution. 1. The quantity to be optimized is represented by the area S of the trapezoid.

2. Denote by x the angle at the known base of the trapezoid. The least possible value of this angle is 60°, then the trapezoid degenerates into an inscribed regular triangle, its side, as is known, being equal to $R\sqrt{3}$ (Fig. 97). On the other hand, x must be less than 120°

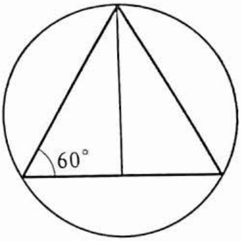


Fig. 97

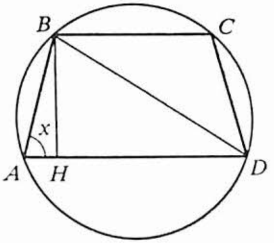


Fig. 98

since the arc subtending the inscribed angle at the base of the trapezoid is less than 240° (Fig. 98).

Thus, the real bounds for the introduced independent variable have been established: $60° \leqslant x < 120°$.

3. Express the area S of the trapezoid $ABCD$ in terms of x and R. We have: $AD = R\sqrt{3}$, $BD = 2R\sin x$, $\angle ABD = \frac{1}{2}\cup AD = 60°$, $\angle BDA = 120° - x$, and $BH = BD\sin(120° - x) = 2R\sin x \times \sin(120° - x)$, BH being the altitude.

We get: $HD = \frac{AD + BC}{2} = BD\cos(120° - x) = 2R\sin x\cos(120° - x)$ and $S = HD \cdot BH = 2R\sin x\cos(120° - x) \cdot 2R\sin x\sin(120° - x) = 2R^2\sin^2 x\sin(240° - 2x)$.

4. Find the greatest value of the function $S = 2R^2\sin^2 x \times \sin(240° - 2x)$ on the half-interval $[60°, 120°)$.

(1) $S' = 2R^2(2\sin x\cos x\sin(240° - 2x) - 2\sin^2 x \times \cos(240° - 2x)) = 4R^2\sin x(\sin(240° - 2x)\cos x - \sin x \times \cos(240° - 2x)) = 4R^2\sin x\sin(240° - 2x - x) = 4R^2\sin x \times \sin(240° - 3x)$.

(2) On the half-interval $[60°, 120°)$, S' vanishes at the point $x = 80°$ only.

(3)

x	60°	80°	120°
S	$\frac{3R^2\sqrt{3}}{4}$	$2R^2\sin^3 80°$	0

In this table, $S(120°)$ is understood as $\lim\limits_{x\to 120°} S$. Compare the values $\frac{3R^2\sqrt{3}}{4}$ and $2R^2\sin^3 80°$. Suppose that $2R^2\sin^3 80° > \frac{3R^2\sqrt{3}}{4}$. Then $\sin^3 80° > \frac{3\sqrt{3}}{8}$, whence $\sin^3 80° > \left(\frac{\sqrt{3}}{2}\right)^3$, that is, $\sin 80° > \frac{\sqrt{3}}{2}$, or $\sin 80° > \sin 60°$. The last inequality and, together with it, our supposition are true. Hence, the function S reaches the greatest value at $x = 80°$.

5. Thus, the greatest area is possessed by the trapezoid with the base angle equal to 80°. It is required to find the lateral side of such a trapezoid. From the triangle ABD (see Fig. 98) we have: $AB = 2R \sin (120° - x)$. For $x = 80°$ we get: $AB = 2R \sin 40°$.

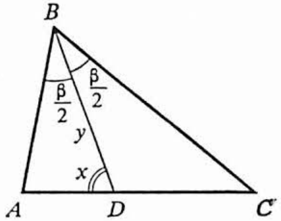


Fig. 99

Example 5. Prove that of all the triangles with a given base and a given vertex angle, an isosceles triangle has the greatest bisector of the vertex angle.

Solution. First Method. 1. The angle bisector BD (Fig. 99) is to be optimized.

2. By the hypothesis, AC and $\angle ABC$ are constant. Set $AC = b$ and $\angle ABC = \beta$. Introduce an independent variable: $x = \angle ADB$.

Find the real bounds of change for x. On the one hand, the angle x, being an exterior angle for the triangle BDC, is greater than any interior angle of this triangle not adjacent to the angle BDA, that is, $x > \frac{\beta}{2}$. On the other hand, from the triangle ABD we conclude that $x < \pi - \frac{\beta}{2}$. Thus, $\frac{\beta}{2} < x < \pi - \frac{\beta}{2}$.

3. Express BD in terms of x, b, and β. Note that $\angle BAD = \pi - x - \frac{\beta}{2}$ and $\angle BCD = x - \frac{\beta}{2}$.

By the law of sines, we get from the triangle ABC: $\frac{AC}{\sin B} = \frac{AB}{\sin C}$, that is, $\frac{b}{\sin\beta} = \frac{AB}{\sin\left(x - \frac{\beta}{2}\right)}$, whence we find that $AB = \frac{b \sin\left(x - \frac{\beta}{2}\right)}{\sin\beta}$.

Analogously, by the law of sines, we get from the triangle ABD: $\frac{AB}{\sin D} = \frac{BD}{\sin A}$, that is, $\frac{AB}{\sin x} = \frac{y}{\sin\left(\pi - x - \frac{\beta}{2}\right)}$, whence we obtain:

$$y = \frac{AB \sin\left(x + \frac{\beta}{2}\right)}{\sin x} = \frac{b \sin\left(x - \frac{\beta}{2}\right) \sin\left(x + \frac{\beta}{2}\right)}{\sin x \sin\beta}$$
$$= \frac{b}{2\sin\beta} \cdot \frac{\cos\beta - \cos 2x}{\sin x}.$$

4. Find the greatest value of the function $y=\frac{b}{2\sin\beta}\cdot\frac{\cos\beta-\cos 2x}{\sin x}$ on the open interval $\left(\frac{\beta}{2},\ \pi-\frac{\beta}{2}\right)$.

(1) $$y'=\frac{b}{2\sin\beta}\cdot\frac{2\sin 2x\sin x-\cos x(\cos\beta-\cos 2x)}{\sin^2 x}=\frac{b}{2\sin\beta}\times\frac{(\cos 2x\cos x+\sin 2x\sin x)+\sin 2x\sin x-\cos\beta\cos x}{\sin^2 x}=\frac{b}{2\sin\beta}\times\frac{\cos x+2\sin^2 x\cos x-\cos\beta\cos x}{\sin^2 x}=\frac{b}{2\sin\beta}\cdot\frac{\cos x\,(1-\cos\beta+2\sin^2 x)}{\sin^2 x}=\frac{b\cos x\left(\sin^2\frac{\beta}{2}+\sin^2 x\right)}{\sin\beta\sin^2 x}.$$

(2) $y'=0$ if $\cos x=0$, that is, for $x=\frac{\pi}{2}$ $\left(\text{the equation } \cos x=0\right.$ has no other solutions on the open interval $\left.\left(\frac{\beta}{2},\ \pi-\frac{\beta}{2}\right)\right)$; y' does not exist if $\sin x=0$, but on the open interval $\left(\frac{\beta}{2},\ \pi-\frac{\beta}{2}\right)$ this equation has no solution.

(3) In order to compile a table for finding the greatest value of the function, let us first of all compute the one-sided limits of the function under consideration for $x\to\frac{\beta}{2}+0$ and for $x\to\pi-\frac{\beta}{2}-0$:

$$\lim_{x\to\frac{\beta}{2}+0}\frac{b(\cos\beta-\cos 2x)}{2\sin\beta\sin x}=\frac{b(\cos\beta-\cos\beta)}{2\sin\beta\sin\frac{\beta}{2}}=0,$$
$$\lim_{x\to\pi-\frac{\beta}{2}-0}\frac{b(\cos\beta-\cos 2x)}{2\sin\beta\sin x}=\frac{b(\cos\beta-\cos(2\pi-\beta))}{2\sin\beta\sin\left(\pi-\frac{\beta}{2}\right)}=0.$$

Now, it is already clear that the greatest value is reached by the function $y(x)$ for $x=\frac{\pi}{2}$. This value is equal to:

$$\frac{b}{2\sin\beta}\cdot\frac{\cos\beta+1}{1}=\frac{b\cdot 2\cos^2\frac{\beta}{2}}{4\sin\frac{\beta}{2}\cos\frac{\beta}{2}}=\frac{b}{2}\cot\frac{\beta}{2}.$$

5. If $x=\frac{\pi}{2}$, then $\angle ADB=90°$. This means that in the triangle ABC, the angle bisector BD is its altitude, hence, the triangle ABC is isosceles Thus, of all the triangles with a given base and a given vertex angle, an isosceles triangle has the greatest bisector of the vertex angle.

Second Method. Let us give a geometrical proof which is considerably briefer and more elegant than the first method.

Circumscribe a circle about the triangle ABC with the angle bisector BD (Fig. 100). The vertices of all the rest of triangles with a given base and a given vertex angle lie on the arc ABC. Let us take an isosceles triangle AB_1C, draw the angle bisector B_1D_1 in it, and prove that $BD < B_1D_1$.

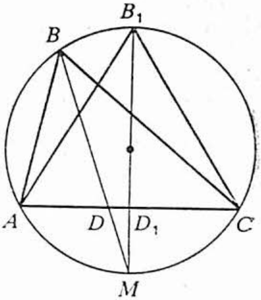


Fig. 100

Extend both angle bisectors BD and B_1D_1 to intersect the circle. Both of them will intersect the circle at one and the same point M which is the midpoint of the arc AC. Since B_1M is a diameter of the circle, we have: $BM < B_1M$. From the triangle DD_1M we conclude that $DM > D_1M$. From these inequalities it follows that $BM - DM < B_1M - D_1M$, that is, $BD < B_1D_1$.

PROBLEMS TO BE SOLVED WITHOUT ASSISTANCE

535. Prove that in a right triangle $\frac{r}{R} \leqslant \sqrt{2} - 1$, where r is the radius of the inscribed circle, and R is the radius of the circumscribed circle.

536. Prove that in an isosceles triangle, the ratio of the radii of the inscribed and circumscribed circles does not exceed $\frac{1}{2}$.

537. Prove that of all the triangles with an equal vertex angle and a constant sum of lateral sides, an isosceles triangle has the least base.

538. Prove that of all the triangles with a given base and a given vertex angle, an isosceles triangle has: (a) the greatest area; (b) the greatest perimeter.

539. Prove that of all the isosceles triangles inscribed in a given circle, an equilateral triangle has: (a) the greatest area; (b) the greatest perimeter.

540. The point A lies between two parallel straight lines l_1 and l_2, is situated at distances a and b from them, respectively, and is the vertex of the right angle of a right triangle ABC; the point B lies on the straight line l_1, and the point C on the straight line l_2. Prove that of all such triangles, the triangle with legs $a\sqrt{2}$ and $b\sqrt{2}$ has the least area.

541. A rectangle is inscribed in a triangle. Prove that the area of the rectangle is not greater than half the area of the triangle.

542. Prove that of all the parallelograms inscribed in a given triangle so that the triangle and parallelograms have a common angle, the greatest area is possessed by the parallelogram whose vertex bisects the side of the triangle opposite to the common angle.

543. The area of a triangle ABC is equal to S and $\angle B = \beta$. Find the least value of: (a) the sum of the sides AB and BC; (b) the side AC; (c) the perimeter of the triangle.

544. Of all the isosceles triangles with a constant length of the median drawn to a lateral side, find the triangle having the greatest area. What is the size of the vertex angle of such a triangle?

545. In the triangle ABC with sides a, b, and c, the sides AB and AC are extended beyond the vertices B and C over the distances AD and AE so that $BD + CE = AC$. Find AD and AE so that the line segment DE has the least length.

546. Taken on the side AC of a triangle ABC is an arbitrary point from which perpendiculars are dropped to the sides AB and BC. What are the least and greatest values of the sum of the lengths of these perpendiculars if it is known that $AB > BC$?

547. In a given right triangle inscribe a rectangle having a common (with the triangle) right angle and a diagonal of the least length.

548. A statue 4 m high is mounted on a column 5.6 m high. At what distance from the column must an observer stand to see the statue at the greatest possible angle if the distance from the ground to the level of his eyes is equal to 1.6 m?

549. The lateral sides and one of the bases of a trapezoid are equal to 15 cm each. For what base will the area of the trapezoid be the greatest?

550. A rectangle of the greatest area is cut away from a right trapezoid with bases a and b and altitude h. Determine this area if: (a) $a = 80$ cm, $b = 60$ cm, and $h = 100$ cm; (b) $a = 24$ cm, $b = 8$ cm, and $h = 12$ cm.

551. The side of a square $ABCD$ is equal to 8 cm. Taken on the sides AB and BC are respective points P and E such that $BP = BE = 3$ cm. On the sides CD and AD find points M and K such that the trapezoid $PEMK$ has the greatest area. What is the greatest value of the area of the trapezoid?

552. In a pentagon $ABCDE$, A, B, and E are right angles, $AB = a$, $AE = b$, $BC = c$, and $DE = m$. In the given pentagon inscribe a rectangle having the greatest area if: (a) $a = 7$ cm, $b = 9$ cm, $c = 3$ cm, and $m = 5$ cm; (b) $a = 7$ cm, $b = 9$ cm, $c = 3$ cm, and $m = 4$ cm.

553. Given on a circle are two points A and B. On the circle, find a point C such that: (a) the product $AC \cdot BC$ is the greatest; (b) the sum $AC + BC$ is the greatest.

554. (a) Of all the sectors with a given perimeter P find the one having the greatest area.

(b) Of all the sectors with a given area S find the one having the least perimeter.

555. The section of a tunnel has the shape of a rectangle completed with a semicircle from above. For what radius of the semicircle will the area of the section be the greatest if the perimeter of the section is equal to P?

556. The distance of the chord AB from the centre O of a circle of radius R is equal to h. In the smaller of the two line segments formed by the chord AB inscribe a rectangle having the greatest area.

557. Inscribed in a circle of radius R is a trapezoid, one base of which is equal to the diameter. Find the greatest area of such a trapezoid.

558. (a) Prove that of all the triangles with a given acute angle at the vertex and a given base, an isosceles triangle has the greatest median drawn to the base.

(b) Prove that of all the triangles with a given obtuse angle at the vertex and a given base, an isosceles triangle has the smallest median drawn to the base.

559. A straight line l is drawn through the vertex B of a given triangle ABC. Perpendiculars are dropped from the points A and C to this line. Prove that the sum of the lengths of these perpendiculars will be the least if the line l is perpendicular to the median BM of the triangle ABC.

560. (a) Of all the isosceles triangles of a given area S find the one in which a circle of the greatest radius can be inscribed. Compute this radius.

(b) About a circle of radius r circumscribe an isosceles triangle of the least area. Find this area.

561. The side of a square $ABCD$ is equal to 6 cm. Taken on the sides AD and AB are points K and P such that $AK = 3$ cm and $AP = 2$ cm. A trapezoid with base KP is inscribed in the square. What is the greatest area of the trapezoid?

562. The chord AB is equal to the radius of a circle. The chord CD is drawn parallel to AB so that the trapezoid $ABCD$ has the greatest area. Find the angular measure of the minor of the arcs subtended by the chord CD.

Chapter 2

SOLID GEOMETRY

SEC. 8. CONSTRUCTING THE REPRESENTATION OF A GIVEN FIGURE

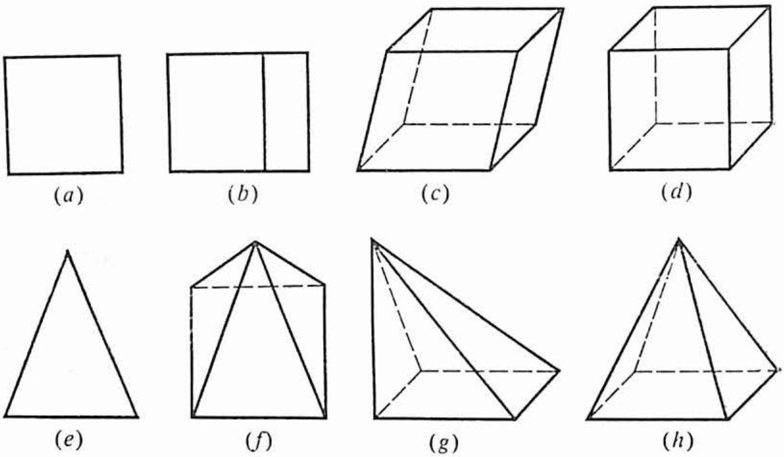
1. In plane geometry, the representation of a given figure Φ_0, called the original, is understood as any figure Φ similar to the figure Φ_0. Thus, if the given figure Φ_0 is a right triangle with legs equal to 5 cm and 30 cm, then it can be represented by any right triangle with legs whose ratio is equal to 5:30, for instance, by a right triangle with legs equal to 1 cm and 6 cm.

Note that in practice it is not always simple to construct the representation of a figure similar to the original. Thus, if for solving a problem it is required to construct the representation of a right triangle given the hypotenuse and bisector of one of the acute angles, then, since neither of the acute angles of the triangle is known, to construct the desired representation of a figure similar to the original, we would have to carry out a certain auxiliary construction, that is, in essence, to solve a new problem. In such cases, it is natural to try to solve the posed problem on an "approximate" drawing, and after the solution is found, to construct a figure similar to the original.

2. In solid geometry, to construct the representation of a given figure is a much more complicated thing.

Descriptive geometry offers various methods of constructing representations, which are developed in detail, various kinds of parallel projection in particular. When solving geometrical problems, the representations of figures are constructed in an arbitrary parallel projection, that is, the position of the original relative to the projection plane and the direction of projection itself relative to this plane are left indeterminate. The possibility of applying such a method of construction of a projective representation follows from the Pohlke-Schwarz theorem, according to which *any plane quadrilateral ABCD together with its diagonals may be taken for a parallel projection of a tetrahedron similar to the tetrahedron* $A_0B_0C_0D_0$ *of an arbitrary form*. From the representation obtained by means of such an arbitrary parallel projection it is impossible to restore the original.

3. Projection drawings made when solving problems at high school must meet the following requirements:



(a) (b) (c) (d)

(e) (f) (g) (h)

Fig. 101

(1) a representation must be *true*, that is, it must be a figure similar to the parallel projection of the original;

(2) a representation must be *descriptive* (*visual*), wherever possible, i. e. it must help visualize the shape of the original (in some cases, this requirement is not observed; this will be explained below);

(3) a representation must be *easily realizable*, that is, the construction rules must be as simple as possible; the abundance of auxiliary constructions impedes comprehension of the contents of a problem.

It is necessary to distinguish clearly between the notions of a true and a descriptive representation.

The truth (correctness) of a representation is a rigorously defined mathematical notion, whereas descriptiveness (visualization) belongs to subjective notions since it is connected with individual perception of a figure being represented.

Thus, all the representations shown in Fig. 101a, b, c, and d are the true representations of a cube, but only the one shown in Fig. 101d is percepted as a visual representation. All the representations of Fig. 101e, f, g, and h are the true representations of a regular quadrangular pyramid, but only one of them is visual (Fig. 101h).

In order for a representation to be true, it is sufficient to construct this representation in accordance with the rules for parallel projection.

4. Tightly connected with the notion of a true representation is the notion of a position completeness of a representation (or, simply, of the completeness of a representation). The representation of a

figure Φ_0 is said to be *complete* if every point A_0 belonging to the figure Φ_0 is given on the projection drawing.

Let us recall briefly how this notion is defined.

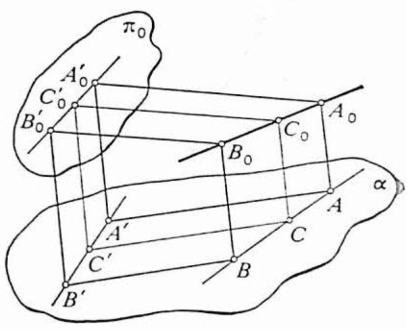


Fig. 102

A plane α on which figures to be studied are projected is called a *projection plane*, or a *plane of representation* (actually, this is the plane of a drawing), and the projection on the plane α itself is said to be *external*. To construct a representation of a certain figure, we may carry out either central or parallel projection. Henceforth, speaking of external projection, we shall mean parallel projection only.

Consider some plane π_0 in space, different from α, and a certain new direction of parallel projection, nonparallel to the plane π_0; the projection in this direction will be called *auxiliary* (parallel). For every point A_0 in space let us construct a point A'_0, i.e. the projection of the point A_0 on the plane π_0 (an auxiliary projection), and then project both points A_0 and A'_0 on the plane α (Fig. 102). We get the points A and A', which are called the *projection* and *secondary projection* of the point A_0, respectively.

The correspondence $A \rightarrow A'$ on a projection plane may be regarded (of course, conventionally) as a certain kind of projection; it is called an *internal parallel projection*, since it is realized inside the plane of representation. It is obvious that whatever the points A_0, B_0, C_0, . . . are in space, $AA' \parallel BB' \parallel CC' \parallel \ldots$ in the plane of representation.

By a projection (representation) of a three-dimensional figure we understand the totality of the projections of all of its points.

In the general case to get a projection of a three-dimensional figure Φ_0 (original), it is not necessary to project each of its points. Thus, if Φ_0 is a polyhedron, then it is bounded by a finite number of faces (plane figures), each face being bounded by a polygonal line whose links are the edges of the polyhedron (line segments). Each edge, in turn, is bounded by a pair of vertices of the polyhedron. Finding the projections of all the vertices of the polyhedron, we thereby determine the projections of all of its edges and faces, that is, the entire projection of the polyhedron.

The point A_0 belonging to the figure Φ_0 is said to be given on the projection drawing (or, simply, a *given point*) if its projection and secondary projection, i.e. a pair of points A and A', are known.

Thus, two pairs of points A, A' and B, B' determine a complete

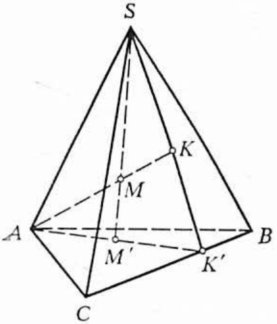


Fig. 103

representation of the straight line A_0B_0 on condition that $AA' \parallel BB'$.

Analogously, if $AA' \parallel BB' \parallel CC'$, then three pairs of points A, A', B, B', and C, C' determine a complete representation of the plane $A_0B_0C_0$.

To justify the completeness of the representation of some figures, it is sometimes expedient to consider the auxiliary central projection of the points A_0, B_0, C_0, . . . of the original figure Φ_0 on the plane π_0. On performing this projection, and then, as usual, the external (parallel) projection of the points A_0 and A_0', B_0 and B_0', C_0 and C_0', . . . on the plane α, we get the projections and secondary projections of the points A_0, B_0, C_0, ... In this case, the correspondence $A \to A'$ is called (of course, conditionally) the *internal central projection*. Obviously, whatever the points A_0, B_0, C_0, . . . in space, the straight lines AA', BB', CC', . . . intersect at one point in the plane of representation (they are said to be concurrent).

Let us now show that if the projections of the vertices of the pyramid $S_0A_0B_0C_0$, i.e. the points S, A, B, and C (without the secondary projections), are given, then the representation of the pyramid will be complete. Indeed, choosing the point S_0 as the centre of auxiliary projection, and the plane $A_0B_0C_0$ as the plane π_0, we can construct the secondary projection, i.e. the point M', for every point M_0 of the pyramid by its projection M. In Fig. 103 this construction is shown for a point M_0 belonging to the plane $S_0A_0K_0'$. In this case, any point of the triangle ABC may be assumed to be the secondary projection of the point S.

The representation of a cone as a figure consisting of an ellipse and a pair of tangents to the ellipse drawn from a certain external point turns out to be complete. To make this sure, we may consider the cone together with the pyramid inscribed in it.

5. If the representation of a figure Φ_0 is complete, then any position problem is solvable on it, that is, any problem on constructing incidences of given figures (for instance, the problem on finding the point of intersection of a given straight line and a given plane).

6. The notion of a true representation is also connected with the notion of its metric definiteness (determinacy).

The representation of a figure Φ_0 is called *metrically determined* if, given this representation, we can (in principle) construct a figure similar to Φ_0.

A complete representation, in the general case, is not yet metrically determined, however, under certain conditions, it may become metrically determined. Thus, if it is indicated that the prism represented in Fig. 101*d* is regular, then its representation will not be metrically determined. But if it is added that the lateral edge of the prism is twice the side of the base, then the metric determinacy of the representation will be guaranteed.

A representation accompanied with some conditions which make it possible to construct a figure similar to the original is said to be *conditional*. These conditions (they are contained in the text of a problem and are usually indicated in the brief notation of the data) are various and depend, specifically, on the figure being represented.

For instance, if a figure Φ_0 is a cube in the original, then it is sufficient to accompany its representation with the condition that a figure Φ is a cube. Indeed, if such a condition is superimposed on the figure Φ, it is possible to construct a figure similar to the original, since all the cubes are similar to one another.

But if a figure Φ_0 is a regular quadrangular pyramid, then it is insufficient to accompany its representation with the condition that a figure Φ is a regular quadrangular pyramid, since in this case we will not be able to construct a figure similar to the original. To make the representation metrically determined here, we have to additionally indicate, say, the ratio of the altitude of the pyramid to the side of the base, or the angle between the lateral edge and the plane of the base, or the angle between the lateral face and the plane of the base.

7. Constructions carried out on a projection drawing may be both positional and metric. *Positional constructions* reproduce the properties of an original retained in external parallel projection. *Metric constructions*, as a rule, reproduce those properties of an original, which are not preserved in parallel projection.

Some constructions, which at first glance seem to be metric, may turn out to be positional in a concrete problem. For example, to construct an altitude of a triangle in the general case is a metric construction, since the property of straight lines to be perpendicular is not preserved in parallel projection. However, if the sides A_0B_0 and B_0C_0 of the triangle $A_0B_0C_0$ are equal in the original, then the altitude B_0D_0 also serves as a median in the original; and the property of a line segment to be a median of a triangle is preserved in parallel projection, therefore, the median BD of the triangle ABC will be the representation of the altitude B_0D_0 of the triangle $A_0B_0C_0$. Thus, in the case under consideration, the construction of the representation of the altitude is positional.

8. To justify the metric determinacy of a complete representation, it is necessary to distinguish between affine and metric properties of figures. Let us recall some of them.

Affine properties (preserved in parallel projection):

(1) the property of a figure to be a point, a straight line, a plane;

(2) the property of figures to have an intersection;

(3) dividing a line segment in a given ratio;

(4) the property of straight lines, planes, a straight line and a plane to be parallel;

(5) the property of a figure to be a triangle, a parallelogram, a trapezoid;

(6) the ratio of the lengths of parallel line segments;

(7) the ratio of the areas of two figures, etc.

Metric properties (not preserved in parallel projection, but preserved in the transformation of similitude):

(1) the property of straight lines, planes, a straight line and a plane to form a certain angle (to be mutually perpendicular in particular);

(2) the ratio of the lengths of nonparallel line segments;

(3) the ratio of the sizes of angles between straight lines (the property of a straight line to be the bisector of an angle in particular);

(4) the ratio of the sizes of dihedral angles;

(5) the ratio of the sizes of angles between straight lines and planes, etc.

For instance, the base of a regular quadrangular prism, which is a square in the original, may be represented as an arbitrary parallelogram, since the ratio of the lengths of the nonparallel sides of a square (equal to unity) and the perpendicularity of its adjacent sides are metric properties, while the parallelism of its opposite sides is an affine property.

In general, an arbitrary parallelogram may represent a parallelogram, a rectangle, a rhombus, and a square.

9. The metric conditions indicated in the text of a problem are naturally not seen in a representation constructed according to the rules of parallel projection. They are usually indicated separately, accompanying the drawing. Supplementing a drawing with metric conditions, we are said to *spend parameters*.

For instance, if a figure Φ_0 is a rhombus $A_0B_0C_0D_0$, and a parallelogram $ABCD$ is its representation, then, adding the words "$ABCD$ is a rhombus" to the drawing, we thereby impose one metric condition on the representation, or in other words, we spend one parameter. Indeed, in the original $A_0B_0 = B_0C_0$, that is, $A_0B_0:B_0C_0 = 1$, but the ratio of the lengths of nonparallel line segments is not preserved in parallel projection, and, consequently, the equality $A_0B_0 = B_0C_0$ expresses one metric property of the original. Analogously, constructing a parallelogram and supplying the drawing with the inscription "a square is represented", we thereby spend two parameters.

If a figure Φ_0 is represented by a certain figure Φ possessing only the affine properties of the original, then no parameters are spent, since the affine properties of the original are preserved in parallel projection.

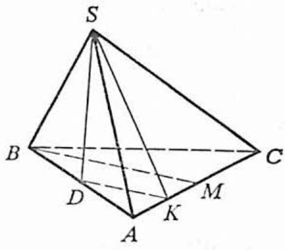


Fig. 104

10. *If, when making a projection drawing, two parameters are spent on representing a plane figure* Φ_0, *then the representation of each point lying in the plane of this figure is thereby determined uniquely* (and thus any further metric constructions in the plane of the figure Φ_0, which may be required to solve the problem, are not allowed to be carried out arbitrarily).

Analogously, *if, when making a projection drawing, five parameters are spent on representing a space figure* Φ_0, *then the representation of each point of this space figure is thereby determined uniquely* (and, consequently, no metric constructions are allowed to be carried out on this drawing arbitrarily).

11. In the process of solving a stereometric problem, we have to carry out various additional constructions, for example, to construct the plane angle of a given dihedral angle, the angle between a given straight line and a given plane, the bisector of a certain angle. When making auxiliary constructions, one should take into account not only the parametric number of the representation (the number of parameters spent), but also the so-called domain of "admissible arrangements".

For instance, the representation of the centre of the circle inscribed in the triangle $A_0B_0C_0$ may not be chosen without regard for the domain of representation of this centre, which, as is known from descriptive geometry, is a part of the plane found inside the triangle whose sides are the midlines of the triangle ABC.

Example 1. One of the lateral faces of a triangular pyramid is perpendicular to the plane of the base. This lateral face and the base of the pyramid are regular triangles. Taking an arbitrary quadrilateral $SBAC$ with its diagonals for the representation of the pyramid, find the parametric number of this representation (Fig. 104).

Solution. We spend two parameters on representing a regular triangle lying in the base of the pyramid by an arbitrary triangle. We spend two more parameters on representing the lateral face, which is a regular triangle in the original, by an arbitrary triangle (one side of this triangle is, naturally, a side of the triangle lying in the base of the pyramid). Finally, assuming that the constructed triangles are the representations of the triangles whose planes are mutually perpendicular in the original, we thereby spend one more parameter. Thus, we have spent all the five parameters on the representation of the given pyramid, that is, its parametric number $p = 5$, and, hence, no other metric constructions are allowed to be carried out arbitrarily on this representation.

Remark. For the sake of brevity, in the phrase "given the representation of a figure", the word "representation" is frequently omitted, and we write simply "given a figure".

Further, let in this example the side of the base of the pyramid be equal to a, and it is required to find the lateral surface area of the pyramid. Omitting for brevity the words "the representation of the pyramid", "the representation of the triangle", and so on, we shall say that given a pyramid $SABC$ whose lateral face SAB and base ABC are regular triangles, the plane SAB being perpendicular to the plane ABC. It is clear that $S_{\Delta SAB} = \frac{a^2\sqrt{3}}{4}$. To find $S_{\Delta SAC}$, we have to find the altitude SK of the triangle SAC.

To solve the problem, it is required to carry out additional constructions, and since the representation is metrically determined (with all the five parameters spent on it), the altitude SK is forbidden to be constructed arbitrarily, that is, taking an arbitrary point K on AC, we have no right to assert: "let SK be perpendicular to AC".

The required additional construction can be carried out in the following way. We draw the median BM of the triangle ABC. Since the latter is a regular triangle, the median BM also serves as the altitude, that is, BM is perpendicular to AC. Analogously, drawing the median SD of the triangle SAB, we have: $SD \perp AB$. It is not difficult to prove that the line segment SD is perpendicular to the plane ABC. We draw DK parallel to BM, then DK is perpendicular to AC. We join the point S to the point K. Since the line segment SD is perpendicular to the plane ABC, DK is the projection of SK on the plane ABC, and, consequently, SK is perpendicular to AC. Now, it remains to perform rather simple computations. We find that $SD = BM = \frac{a\sqrt{3}}{2}$, $DK = \frac{BM}{2} = \frac{a\sqrt{3}}{4}$, and $SK = \frac{a\sqrt{15}}{4}$. Thus, $S_{\text{lat}} = \frac{a^2\sqrt{3}}{4}(1+\sqrt{5})$.

Example 2. The perpendicular BK is dropped to the plane ABC at the vertex B of the equilateral triangle ABC, BK being equal to AB. Find the tangent of the acute angle between the straight lines AK and BC (Fig. 105).

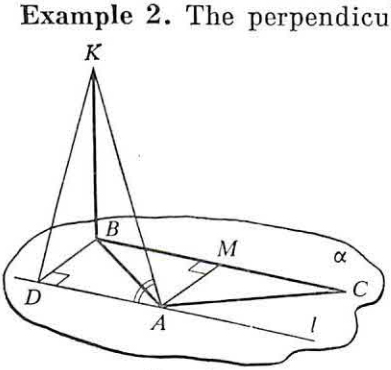


Fig. 105

Solution. We construct the representation of the given figure and find its parametric number. Assuming an arbitrary triangle ABC to be the representation of the equilateral triangle, we spend two parameters. Assuming the line segment

BK to be the representation of the perpendicular to the plane ABC, we also spend two parameters. Finally, assuming that $BK:AB = 1:1$, we spend one more parameter. Hence, the parametric number $p = 5$. Thus, the given representation is metrically determined, and any other metric constructions on it are inadmissible.

We then carry out the following additional constructions: in the plane ABC, through the point A draw a straight line l parallel to BC. Then, the angle between the straight lines AK and BC is equal to the angle between the straight lines AK and l. To find the angle between the straight lines AK and l, it is expedient to include this angle in some right triangle. The straight line DK, which is perpendicular to l, can be constructed as an inclined line whose projection is perpendicular to the straight line l. To draw this projection (the straight line BD), we take advantage of the fact that the triangle ABC is equilateral, and, consequently, its median AM is perpendicular to the line segment BC. Thus, having constructed the median AM of the triangle ABC, then BD parallel to AM, and the line segment DK, we get a right triangle ADK. The ratio $DK:AD = \tan \angle DAK$ is the required one. Setting $AB = a$, we find that $BK = a$, $BD = \frac{a\sqrt{3}}{2}$, $DK = \frac{a\sqrt{7}}{2}$, and $AD = BM = \frac{a}{2}$. Thus, $\tan \angle DAK = \sqrt{7}$, and, consequently, the tangent of the angle between the straight lines AK and BC is equal to $\sqrt{7}$.

Example 3. One of the legs of a right isosceles triangle lies in the plane α, and the other forms an angle equal to 45° with it. Find the representation of the given figure, its parametric number, and then the angle formed by the hypotenuse and the plane α (Fig. 106).

Solution. Assuming an arbitrary triangle ABC to be the representation of the right triangle, we spend one parameter. Assuming that AC and BC are the representations of congruent line segments, we spend one more parameter. Finally, assuming that BC is the representation of the line making an angle equal to 45° with the plane α, we also spend one more parameter. Thus, the parametric number $p = 3$. Therefore, there are some more vacant parameters for further metric constructions on this representation.

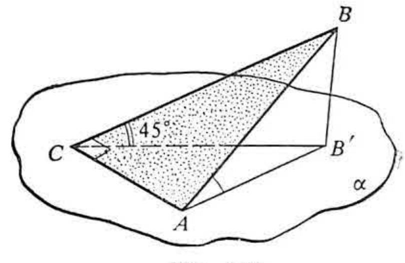


Fig. 106

Let us now determine the angle formed by the hypotenuse AB and the plane α. We take some point B' in the plane α and assume that BB' represents the perpendicular to the plane α. The last two vacant parameters are also spent. The representation becomes metrically determined, and, therefore, any further metric constructions are not allowed to be carried out arbitrarily on the obtained representation. However, this is not required in our case. We then construct $B'C$ and $B'A$. Setting $AC = a$, we find that $BC = a$, $BB' = \frac{a\sqrt{2}}{2}$, and $AB = a\sqrt{2}$. Thus, in the right triangle ABB' we have: BB': $AB = 1{:}2$, that is, $\angle BAB' = 30°$. But $BB' \perp \alpha$, i.e. $\angle BAB'$ is the angle formed by the hypotenuse AB and the plane α.

Remark. To get a visual representation, we chose arbitrarily the point B', which belongs to the plane α, so that the straight line BB' was parallel to the margin of the page. This representation prompts the idea that B_0B_0' is perpennicular to α_0 in the original. Note that the arbitrariness in choosing point B' is bot without bounds. For instance, the straight line BB' must not turn out to de parallel to the straight line AC (in this case, the straight line BB' would be the representation of the straight line B_0B_0' parallel to the plane α_0).

Example 4. In a regular quadrangular pyramid, the angle between two adjacent lateral faces is equal to 2α. Find the angle formed by a lateral edge of the pyramid and the plane of its base (Fig. 107).

Solution. It is clear that the figure $SABCD$ is a complete representation of the given pyramid. Let us count the parametric number of this representation. Two parameters are spent on representing the square serving as the base of the given pyramid by an arbitrary parallelogram $ABCD$. The other two parameters are spent on the altitude of the pyramid represented by the line segment SO, where O is the point of intersection of the diagonals of the parallelogram $ABCD$, since it is assumed that the line segment SO is the representation of the perpendicular to the plane ABC. Thus, the parametric number $p = 4$, that is, only one vacant parameter is left for making further metric constructions that may be required in the process of solving the problem.

To solve the given problem, let us construct the plane angle of the given dihedral angle on this representation. This can be done, for instance, in the following way: we take an arbitrary point M on the edge SC and assume the line segment OM to be the representation of the perpendicular dropped from the point O_0 to the edge S_0C_0. Joining now the point M to the points B and D, we get the angle BMD, which is, as is easy to show, the representation of the plane angle (of the dihedral angle S_0C_0 in the original).

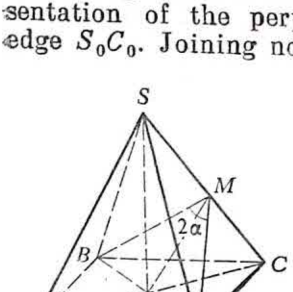


Fig. 107

We may now introduce the notation 2α into the drawing. Since the line segment SO is the representation of the altitude of the pyramid, the line segment OD is the representation of the projection of the lateral edge on the plane of the base of the pyramid, and, therefore, the angle SDO is the representation of the desired angle. It is easy to compute that this angle is equal to arcsin $(\cot \alpha)$, where $45° < \alpha < 90°$.

12. When solving some stereometric problems, not complete, metrically determined, but simplified representations turn out to be more suitable. For instance, when solving problems on combinations of polyhedrons and round solids, after having chosen a section suitable for constructing a combination of solids, we can show on this representation the figure similar to the original and obtained in the section.

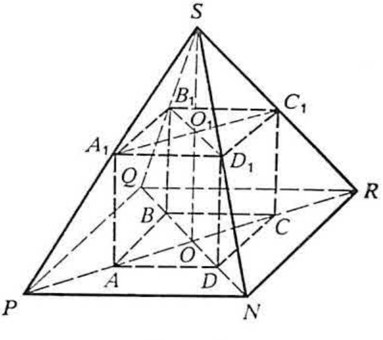


Fig. 108

Thus, instead of the representation of a regular quadrangular pyramid and a ball inscribed in it, we may give the representation of the figure obtained in the section of this combination of solids by the plane passing through the altitude of the pyramid parallel to the side of the base. This section is an isosceles triangle whose lateral sides are the slant heights of the pyramid. Since the centre of the ball lies on the altitude of the pyramid, the section of the ball by this plane will be the circumference of the great circle. Since the ball touches the faces of the pyramid, its centre lies in the bisecting plane of the planes of its opposite faces. But the bisecting plane intersects two other lateral faces along their slant heights.

Thus, the circle obtained in the section touches the lateral sides of the isosceles triangle. Analogously, this circle also touches the base of the triangle. Hence, the section turns out to be an isosceles triangle with the circle inscribed in it.

13. Sometimes, when making a drawing, it is expedient to carry out the required construction "from the end".

Example 5. A cube is inscribed in a regular quadrangular pyramid so that its four vertices lie in the base of the pyramid, and the other four on the lateral edges. Construct the representation of the given figure.

Solution. Let $ABCDA_1B_1C_1D_1$ be the representation of the cube (Fig. 108). First of all, we find the centres of the bases of the cube (points O and O_1). Then we draw the straight line OO_1 and take the point S (outside the cube). Further, we draw the straight lines SA_1, SB_1, SC_1, SD_1 and find the points P, Q, R, N, i.e. the vertices of the base of the pyramid. Then $SPQRN$ is the representation of the given pyramid.

14. We should like to close this section with the following remark. If a problem involves a figure whose construction demands no more than five parameters, then, in the process of solution, the construc-

tion of the relevant representation is not described (but carried out necessarily). In such cases, we make sure that the constructed representation is complete and count its parametric number. The latter procedure is quite necessary since, in the process of solution, additional constructions of metric character are required, which with some vacant parameters at hand can be carried out arbitrarily (with due regard for concrete limitations in carrying out these constructions).

If a problem deals with a figure whose construction requires more than five parameters, then the part of the construction of a representation whose performance requires five parameters is usually not described, all further constructions are necessarily being described and carried out according to the rules for parallel projection.

If it is stipulated in a problem that certain elements of a given figure are required to be constructed (for instance, a section in a given pyramid), then the construction of the representation of these elements of the given figure is described necessarily. Of course, having carried out this or that construction, we count the parametric number of the obtained representation (such examples are considered in Sec. 12). In some cases, the description of this construction is carried out according to the complete scheme for solving a construction problem (analysis, construction, proof, investigation), whereas in other cases the description is confined to individual steps of this scheme, for instance, to construction and proof or construction and investigation.

SEC. 9. GEOMETRICAL CONSTRUCTIONS IN SPACE

I. Simplest Constructions in Space

Example 1. Through a given point A draw a plane parallel to a given plane P (Fig. 109).

Solution. Analysis. Let Q be the required plane. In the plane Q construct two distinct straight lines l and m passing through the point A, and in the plane P take an arbitrary point A_1. Construct the planes L and M passing through the point A_1 and the straight line l, and through the point A_1 and the straight line m, respectively. Since the straight line l lies in the plane Q, which is parallel to P, we have: $l \parallel P$. But then the plane L intersects the plane P along a straight line l_1, which is parallel to l. Analogously, the plane M intersects the plane P along a straight line m_1, which is parallel to m.

Since Q is determined uniquely by the straight lines l and m, the problem is reduced to constructing the straight lines l and m passing through the point A parallel to the plane P.

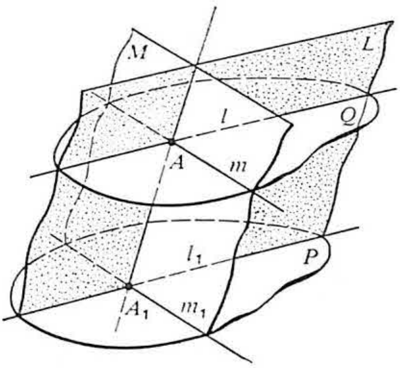


Fig. 109

Construction. (1) In the plane P take an arbitrary point A_1. (2) Through the point A_1 in the plane P draw arbitrary straight lines l_1 and m_1 ($l_1 \nparallel m_1$). (3) Construct the planes L and M. (4) Through the point A in the plane L draw a straight line l parallel to l_1, and in the plane M a straight line m parallel to m_1. (5) Draw the plane Q through the straight lines l and m.

Proof. Since, by construction, l is parallel to l_1, and the straight line l_1 lies in the plane P, we have: $l \parallel P$. Analogously: $m \parallel P$. Then the plane Q passing through the straight lines l and m is parallel to the plane P and also passes through the point A. Thus, Q is the required plane.

Investigation. The problem has the unique solution. Indeed, let us assume the contrary, that is, let us assume that there is another plane Q_1, not coincident with Q, which is parallel to the plane P and passes through the point A. Then the plane L intersects the plane Q_1 along a straight line l_2 parallel to l. But this means that in the plane L, through the point A, two straight lines l and l_2 pass parallel to the straight line l_1, which contradicts the parallelism axiom. The obtained contradiction shows that the problem has the only solution. In a particular case, when the given point A lies in the plane P, the planes Q and P coincide.

II. Loci of Points

Example 2. Find the locus of points in space, which are equidistant from given intersecting planes P_1 and P_2 (Fig. 110).

Solution. (1) Let M be a point equidistant from the planes P_1 and P_2. Construct the plane Q passing through the point M perpendicular to a, i.e. the line of intersection of the planes P_1 and P_2. To this end, drop the perpendicular ML from the point M to the straight line a, and then in the plane P_1, through the point L, draw a straight line l_1 perpendicular to a. The plane Q, determined by the straight lines LM and l_1, is perpendicular to the straight line a. Now construct l_2, i.e. the line of intersection of the planes P_2 and Q. Then l_2 is perpendicular to a. Further, drop perpendiculars from the point M: MM_1 to the straight line l_1 and MM_2 to the straight line l_2. Since $a \perp l_1$ and $a \perp LM$, we have: $a \perp MM_1$, or $MM_1 \perp a$. But the fact that $MM_1 \perp a$ and $MM_1 \perp l_1$ implies that $MM_1 \perp P_1$, that is, the length of the line segment MM_1 is the distance from the

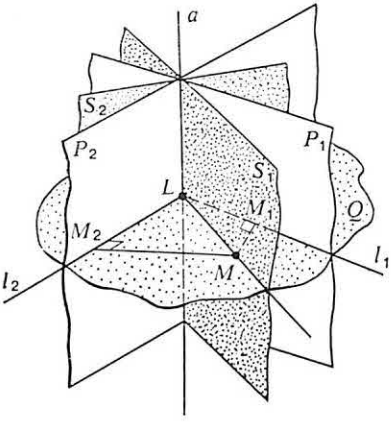


Fig. 110

point M to the plane P_1. Analogously, the length of the line segment MM_2 is the distance from the point M to the plane P_2. Then in the right triangles MM_1L and MM_2L we have: $MM_1 = MM_2$, and LM is their common hypotenuse. Therefore, the triangles MM_1L and MM_2L are congruent, and this means that $\angle MLM_1 = \angle MLM_2$. But, by construction, $\angle MLM_1$ is the plane angle of the dihedral angle M_1aM, and, analogously, $\angle MLM_2$ is the plane angle of the dihedral angle M_2aM. Thus, the dihedral angles M_1aM and M_2aM are congruent, that is, the half-plane given by the point M and the straight line a is the bisecting half-plane of the dihedral angle M_1aM_2. Thus, if the point M is equidistant from the two intersecting planes P_1 and P_2, then it belongs to the bisecting half-plane of the dihedral angle formed by the planes P_1 and P_2.

But the intersecting planes P_1 and P_2 form four dihedral angles, and, consequently, the locus of all the points M is a pair of planes S_1 and S_2, where S_1 is the union of two bisecting half-planes of one pair of vertical angles formed by the planes P_1 and P_2, and S_2 is the union of two bisecting half-planes of the other pair of vertical angles formed by P_1 and P_2.

(2) Let the point M belong to some bisecting half-plane of the dihedral angles formed at the intersection of P_1 and P_2. Construct the plane Q passing through M perpendicular to a, for which purpose drop the perpendicular ML from the point M to a, and then through the point L draw l_1 perpendicular to a.

The plane Q is determined by the straight lines l_1 and LM. Further, in the plane Q construct a straight line l_2 passing through the point L perpendicular to a, and from the point M drop two perpendiculars $MM_1 \perp l_1$ and $MM_2 \perp l_2$. Considering the right triangles MLM_1 and MLM_2 in which $\angle MLM_1 = \angle MLM_2$ (since the ray LM is the bisector of the plane angle M_1LM_2, and the line segment LM is their common hypotenuse), we conclude that $MM_1 = MM_2$. Thus, if the point M belongs to the bisecting half-plane of one of the four dihedral angles formed by the planes P_1 and P_2, then it is equidistant from these planes.

Thus, the sought-for locus of points is the union of the bisecting half-planes of the four dihedral angles made by the planes P_1 and P_2, that is, the union of the planes S_1 and S_2.

Remark. We could show that S_1 is perpendicular to S_2, but we leave this for the reader to do independently.

III. Applications of Certain Loci of Points and Straight Lines

Example 3. Construct the locus of points belonging to a given plane P and equidistant from given points A and B not lying in the plane (Fig. 111).

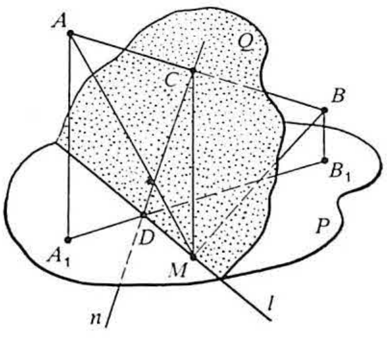


Fig. 111

Solution. Analysis. Let M belong to the required locus of points. Since $AM = BM$, the point M belongs to the plane Q passing through the point C, i.e. the midpoint of the line segment AB, perpendicular to the line segment AB. Since, in addition, the point M lies in the plane P, the sought-for locus of points is the line of intersection of the planes P and Q. We denote this line by l. The facts that AB is perpendicular to Q and l lies in the plane Q imply that AB is perpendicular to l. Let $AA_1 \perp P$, $BB_1 \perp P$, and l intersect A_1B_1 at the point D. Then AB is perpendicular to CD, and, consequently, A_1B_1 is also perpendicular to l, or $l \perp A_1B_1$.

Thus, the problem can be reduced to constructing the straight line l passing in the plane P through the point D perpendicular to the line A_1B_1.

Construction. (1) Construct $AA_1 \perp P$ and $BB_1 \perp P$. (2) Construct A_1B_1. (3) Find the point C, i.e. the midpoint of the line segment AB. (4) In the plane passing through the points A, A_1, and B draw through the point C a straight line $n \perp AB$. (5) Find the point D at which the straight lines n and A_1B_1 intersect. (6) In the plane P construct a straight line $l \perp A_1B_1$ passing through the point D. (7) Construct the plane Q passing through the straight lines l and n.

Proof. If the point M lies on the straight line l, then it also lies in the plane P. Join the point M to the points A, B, and C. Since AB is perpendicular to Q, and the straight line CM lies in the plane Q, we have: $AB \perp CM$. In addition, $AC = BC$. Then the triangles ACM and BCM are congruent (by two legs), and, therefore, $AM = BM$. Thus, the straight line l is the required locus of points.

Investigation. The existence of a solution depends on the mutual position of the straight line AB and the plane Q.

(1) If $AB \perp P$, then, since $AB \perp Q$, it turns out that Q is parallel to P, that is, there is no solution in this case.

(2) If AB is not perpendicular to P, then the line l of intersection of the planes P and Q is the required locus of points.

IV. Constructions on Representations

For the sake of brevity, let us agree to omit to word "representation" everywhere while formulating both the examples of this item and the problems for independent solution. For instance, instead of writing: "the triangle ABC serves as a representation of the triangle $A_0B_0C_0$, in which $A_0B_0:B_0C_0 = 2:3$", we shall write more briefly: "in the triangle ABC, $AB:BC = 2:3$". Analogously, instead of the words "let us construct the representation of the bisector of the angle $A_0B_0C_0$", we shall write: "let us construct the bisector of the angle ABC".

Let us also agree to use zero as a subscript for denoting the original figure throughout this item. For instance, by stating that the median A_0M_0 is constructed in the triangle $A_0B_0C_0$, we mean that this construction is carried out in the original. In this case, the representation will show the triangle ABC with the median AM constructed in it.

Let us consider a number of examples.

Example 4. In the triangle ABC, $AB:BC = 2:3$. Construct the bisector of the angle ABC (Fig. 112).

Solution. First Method. If the line segment BD is the representation of the bisector B_0D_0 of the angle $A_0B_0C_0$, then $AD:DC = AB:BC$, that is, $AD:DC = 2:3$. Thus, constructing a point D on the side AC such that $AD:DC = 2:3$, we get BD, i.e. the representation of the desired angle bisector. (Figure 112 shows the construction of the point D by means of an auxiliary ray AC_1 on which $AD_1:D_1C_1 = 2:3$.)

Second Method. Let us find on the sides AB and BC points K and L, respectively, such that $BK = \frac{1}{2} AB$ and $BL = \frac{1}{3} BC$ (see Fig. 112). Then the triangle BLK is the representation of the isosceles triangle $B_0L_0K_0$, and, therefore, the median BN of the triangle BLK is the representation of the bisector of the angle $A_0B_0C_0$.

Third Method. We find, as in the second method, the points K and L and then construct $KM \parallel BC$ and $LM \parallel AB$ (see Fig. 112). Then the parallelogram $BLMK$ is the rhombus, and the line segment BM will be the representation of the diagonal of the rhombus, which, as is known, bisects the angle of the rhombus. Thus, the half-line B_0M_0 contains the bisector of the angle $A_0B_0C_0$. Having found the point D, i.e. the intersection of the half-line BM and the side AC of the triangle ABC, we get the line segment BD, i.e. the representation of the desired angle bisector.

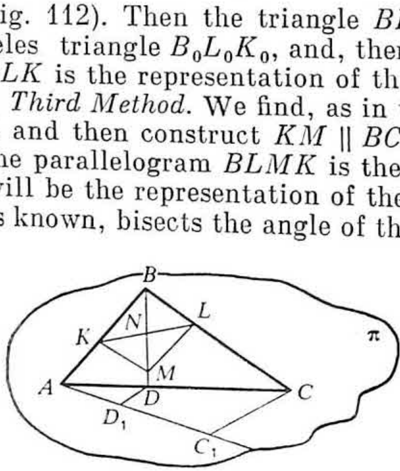


Fig. 112

Remark. Only one parameter has been spent on the representation of the

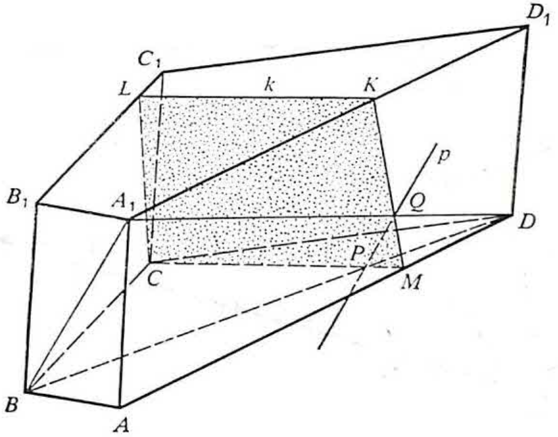


Fig. 113

triangle $A_0B_0C_0$ ($A_0B_0:B_0C_0 = 2:3$). Therefore, we are still free to a certain extent to carry out some metric constructions (there is one parameter in store). But when constructing the representation of the bisector of the angle $A_0B_0C_0$, on having taken an arbitrary point D on the line segment AC, we have no right to say that the line segment BD is the representation of the bisector of the angle $A_0B_0C_0$, whereas when representing, for instance, the bisector of the angle $B_0A_0C_0$, on having taken an arbitrary point E on the line segment BC, we may assume the line segment AE to be the representation of the bisector of the angle $B_0A_0C_0$.

Example 5. A point P is taken on the diagonal BD of the base of a quadrangular prism $ABCDA_1B_1C_1D_1$. Construct the section of the prism by the plane passing through the straight line CP parallel to the straight line A_1B (Fig. 113).

Solution. Let α be the plane of the required section. Since A_1B is parallel to α, the plane β, passing through the straight line A_1B nonparallel to α, intersects α along a straight line parallel to A_1B. We draw a plane BA_1D through the straight line A_1B and some point P on the straight line CP and denote it by β. Further, we find the line along which β intersects the plane AA_1D_1. It is clear that this will be the straight line A_1D, since the two points A_1 and D lie both in the plane β and in the plane AA_1D_1. Drawing a straight line p parallel to A_1B through the point P in the plane β, we find the point Q of intersection of the straight lines p and A_1D.

The plane of the section α is now specified by the straight lines CM (where M is the point of intersection of the straight lines CP and AD) and PQ. Since the two points M and Q lie in both the plane α and the plane AA_1D_1, these planes intersect along the line MQ. We then find K, i.e. the point of intersection of the straight lines MQ and A_1D_1. Since the planes ABC and $A_1B_1C_1$ are parallel, the lines along

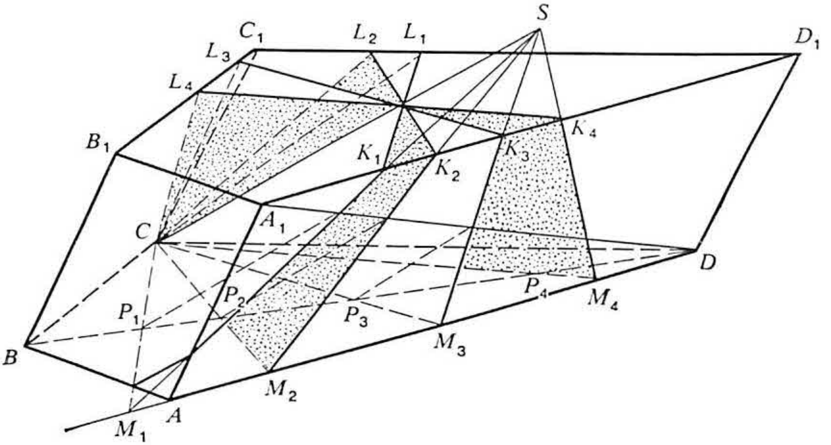


Fig. 114

which these planes intersect with the cutting plane α are also parallel. Thus, in the plane $A_1B_1C_1$, through the point K, we draw a straight line k parallel to CM and join the point L of intersection of the straight lines k and B_1C_1 to the point C. The quadrilateral $CMKL$ is the desired section.

Investigation. Since the point A_1 does not lie in the plane ABC, A_1B and CP are skew lines, and, consequently, the cutting plane α is unique. Depending on the position of the point P on the diagonal BD and the kind of the given prism, the section can have a shape different from the one shown in Fig. 113. Some shapes of section are shown in Fig. 114.

Example 6. In a rectangular parallelepiped $ABCDA_1B_1C_1D_1$, $AB{:}AD{:}AA_1 = 1{:}2{:}2$. Drop a perpendicular from the point A_1 to the diagonal B_1D (Fig. 115).

Solution. Note that the representation of the given parallelepiped is complete and metrically determined. Indeed, assuming the parallelogram $ABCD$ to be the representation of the rectangle, we spend one parameter. Assuming then that arbitrary line segments AB and AD are such that $AB{:}AD = 1{:}2$, we spend one more parameter. Further, assuming that the line segment AA_1 is the representation of a perpendicular to the plane ABC, we spend two parameters. Finally, assuming that $AD{:}AA_1 = 1{:}1$, we spend one more parameter. Thus, we have spent five parameters on the representation of the given parallelepiped, that is, its

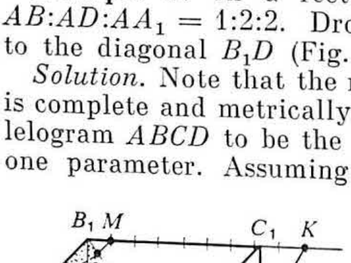


Fig. 115

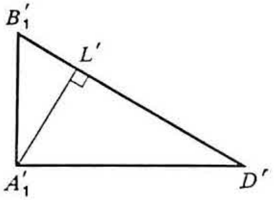


Fig. 116

representation is metrically determined. But then, taking an arbitrary point L on B_1D, we are not allowed to assume A_1L to be the representation of the perpendicular to B_1D.

We are going to give two methods of constructing $A_1L \perp B_1D$.

First Method. Let us find certain relationships, which can be used for the required construction. Setting $AB = a$, we get: $AD = 2a$ and $AA_1 = 2a$. Then we find from the right triangle AA_1D that $A_1D = 2a\sqrt{2}$, and we find from the right triangle A_1B_1D that $B_1D = 3a$. Let us now compute in what ratio the required perpendicular A_1L will divide the side B_1D of the right triangle A_1B_1D. We have from the similarity of the right triangles A_1B_1L and A_1B_1D: $\frac{B_1L}{A_1B_1} = \frac{A_1B_1}{B_1D}$, whence $B_1L = \frac{a}{3}$. Thus, $B_1L{:}B_1D = \frac{a}{3}{:}3a$, that is, $B_1L{:}B_1D = 1{:}9$. Since parallel projection preserves the ratio of the lengths of parallel line segments, on the constructed representation, as in the original, the foot L of the desired perpendicular must divide the line segment B_1D in the ratio $B_1L{:}B_1D = 1{:}9$.

Now, let us carry out the relevant construction. On some ray whose origin is the point B_1, e.g. on the ray B_1C_1, we lay off nine equal line segments and through the point M, i.e. the end point of the first line segment, draw a straight line parallel to KD. Then L, i.e. the point of intersection of the constructed straight line and B_1D, will be the foot of the desired perpendicular.

Second Method. Let us construct the triangle $A_1'B_1'D'$ similar to the original (preimage) of the triangle A_1B_1D (Fig. 116), that is, the triangle with a right angle at the vertex A_1' and the ratio of the legs $A_1'B_1'{:}A_1'D' = 1{:}2\sqrt{2}$.

(This ratio follows from the fact that, by the hypothesis, $AB{:}AD{:}AA_1 = 1{:}2{:}2$, that is, if $AB = a$, then $AD = AA_1 = 2a$ and $A_1D = 2a\sqrt{2}$. Consequently, $AB{:}A_1D = a{:}2a\sqrt{2}$, or $AB{:}A_1D = 1{:}2\sqrt{2}$.)

In the triangle $A_1'B_1'D'$ draw the perpendicular $A_1'L'$ from the vertex A_1' to the side $B_1'D'$. The line segment $B_1'D'$ will be divided by the point L' in a certain ratio: $B_1'L'{:}B_1'D'$. The line segment B_1D will be divided by the point L in the same ratio.

As is clear, when constructing the point L by the second method, there is no need to compute the ratio $B_1L{:}B_1D$.

Example 7. The base of a right parallelepiped $ABCDA_1B_1C_1D_1$ is a parallelogram with an acute angle of 60° and the ratio of sides $AB{:}AD = 3{:}4$. Construct the plane passing through the edge AA_1 perpendicular to the diagonal plane B_1BDD_1 (Fig. 117).

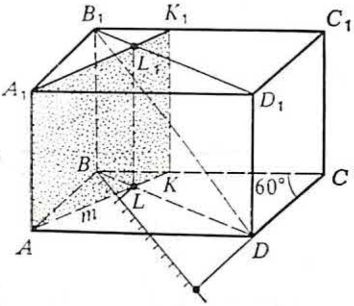


Fig. 117

Solution. It is obvious that if a perpendicular m is dropped from a point on the straight line AA_1 to the plane BDD_1, then the plane specified by the straight lines AA_1 and m is perpendicular to the plane B_1BDD_1, thus being the desired plane.

Suppose that the straight line m passes through the point A. Since the given parallelepiped is a right one, the straight line BB_1 is perpendicular to any straight line lying in the plane of the base, i.e. it is advisable to draw the straight line m in the plane of the base. Furthermore, since the straight line m must be perpendicular to the diagonal plane B_1BDD_1, m must be perpendicular to BD. Thus, the problem on constructing the plane perpendicular to the plane B_1BDD_1 is practically reduced to the problem on constructing the straight line m passing through the point A perpendicular to BD.

May we take an arbitrary point L on the straight line BD and assume that AL is perpendicular to BD? To answer this question, we have to determine the parametric number p of the representation of the base of the parallelepiped. If $p = 2$, that is, the representation is metrically determined, we are not allowed to take the point L arbitrarily. Thus, let us count the parameters spent on the representation. Assuming the quadrilateral $ABCD$ to be the representation of the parallelogram, we spend no parameter, since in parallel projection, parallelism is preserved, whereas the ratio of the lengths of nonparallel line segments is not preserved, that is, assuming that the arbitrarily taken line segments AB and AD are just such that $AB{:}AD = 3{:}4$, we spend one parameter Furthermore, since the size of angles does change, assuming that the angle BCD is the representation of the angle equal to 60°, we spend one more parameter. Thus, the parametric number of the representation of the base of the parallelepiped $p = 2$, that is, no new assumptions of metric character (that is, assumptions concerning the presence of properties not preserved in parallel projection) should be made. Hence, the point L, i.e. the foot of the perpendicular dropped from the point A to the straight line BD, is forbidden to be taken arbitrarily. It can be constructed on having computed the ratio $BL{:}BD$.

Carrying out these computations, we set $AB = 3a$, then $AD = 4a$. Further, by the law of cosines, $BD = a\sqrt{13}$, and from the equation $AB^2 - BL^2 = AD^2 - (BD - BL)^2$ we find that $BL = \frac{3a}{\sqrt{13}}$. Then $BL{:}BD = 3{:}13$. On having constructed the point L

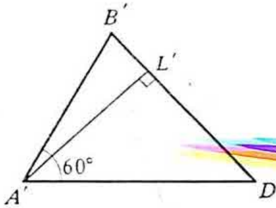


Fig. 118

(with the aid of an auxiliary ray on which thirteen equal line segments are laid off), we then draw through this point a straight line parallel to BB_1, and, finally, the desired plane AKK_1A_1.

Remark. The construction $AL \perp BD$ could be carried out as in the preceding example (see the second method) with the aid of an auxiliary drawing (Fig. 118). Namely, we construct the triangle $A'B'D'$ similar to the original (preimage) of the triangle ABD, that is, the triangle with the angle at the vertex A' equal to 60° and the ratio of the sides $A'B':A'D' = 3:4$. In this triangle we drop the perpendicular $A'L'$ from the vertex A' to the side $B'D'$. The line segment $B'D'$ is divided by the point L' in a certain ratio $B'L':B'D'$. The line segment BD is divided by the point L in the same ratio.

Example 8. The figure $ABCDA_1B_1C_1D_1$ is a cube. Constructed in it is a section passing through the vertex B and points P and Q, i.e. the midpoints of the edges A_1B_1 and B_1C_1, respectively. Drop a perpendicular from the vertex D to this plane (Fig. 119).

Solution. Since the representation of the cube is complete and metrically determined (five parameters have been spent on it), on having taken an arbitrary point N in the plane PQB, we are not allowed to say that DN is perpendicular to the plane PQB. To construct the desired perpendicular, we first construct the diagonal plane BB_1D_1 of the cube. The planes BB_1D_1 and PQB are mutually perpendicular. Indeed, PQ is the midline of the triangle $A_1B_1C_1$, therefore, $PQ \parallel A_1C_1$, and, hence, $PQ \perp B_1D_1$. In addition, since $BB_1 \perp A_1B_1$ and $BB_1 \perp B_1C_1$, we have: $BB_1 \perp PQ$, or $PQ \perp BB_1$. Thus, $PQ \perp B_1D_1$ and $PQ \perp BB_1$, i.e. PQ is perpendicular to the plane BB_1D_1, and then the plane PQB passing through PQ is also perpendicular to the plane BB_1D_1. And since the planes BB_1D_1 and PQB are mutually perpendicular, the perpendicular dropped from the point D to the line of intersection of these planes is just the required perpendicular. The line of intersection of the planes BB_1D_1 and PQB is found easily—this is the line BK passing through common points B and K of these planes.

Now, let us carry out all necessary constructions in the diagonal plane BB_1D_1. As in the preceding example, this can be done by two methods.

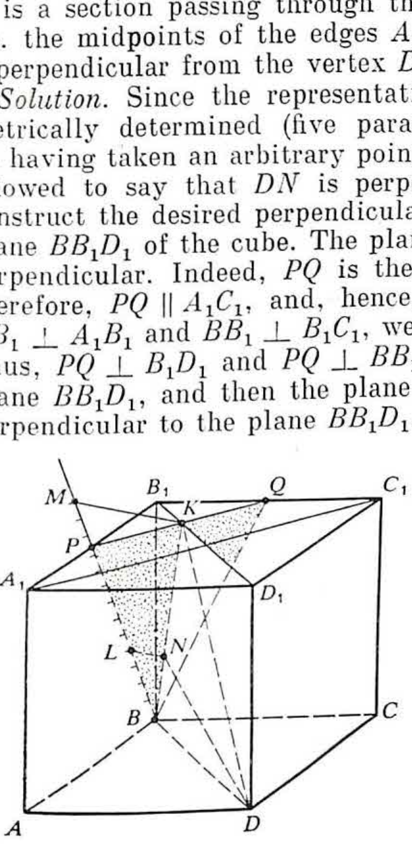


Fig. 119

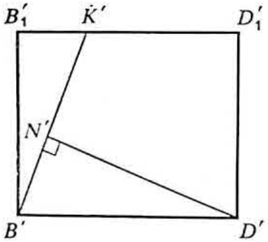


Fig. 120

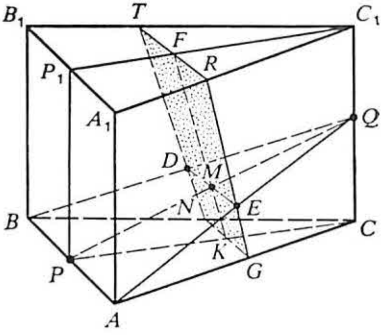


Fig. 121

First Method. Let $AB = a$. We compute the sides of the triangle BKD: $BD = a\sqrt{2}$, $BK = \frac{3a\sqrt{2}}{4}$ from the triangle BB_1K, and $DK = \frac{a\sqrt{34}}{4}$ from the right triangle DD_1K. If $DN \perp BK$, then $DB^2 - BN^2 = DK^2 - KN^2$, or $DB^2 - BN^2 = DK^2 - (BK - BN)^2$, whence $BN = \frac{a\sqrt{2}}{3}$. Thus, $BN:BK = \frac{a\sqrt{2}}{3}:\frac{3a\sqrt{2}}{4}$, that is, $BN:BK = 4:9$. Since parallel projection preserves the ratio of the lengths of parallel line segments, on the representation, the same as in the original, the point N, i.e. the foot of the desired perpendicular, divides BK in the ratio $BN:BK = 4:9$. The construction of the point N by means of an auxiliary ray BP is shown in Fig. 119.

Second Method. Construct a rectangle similar to the original of the rectangle BB_1D_1D, that is, the rectangle $B'B_1'D_1'D'$ with the ratio of sides $B'B_1':B_1'D_1' = 1:\sqrt{2}$ (Fig. 120). Further, on the side $B_1'D_1'$ construct a point K' such that $B_1'K':B_1'D_1' = 1:4$ and join the points B' and K'. Then drop the perpendicular $D'N'$ from the point D' to the side $B'K'$ of the triangle $B'K'D'$. The point N' divides the line segment $B'K'$ in a certain ratio. The point N divides the line segment BK in the same ratio.

Example 9. Taken on the edges AB and CC_1 of a regular triangular prism $ABCA_1B_1C_1$ in which $AA_1:AB = 1:2$ are points P and Q, i.e. the midpoints of these edges. Construct the locus of points lying on the surface of the prism and equidistant from the points P and Q (Fig. 121).

Solution. The representation of the given prism is complete and metrically determined. Indeed, assuming an arbitrary triangle ABC to be the representation of a regular triangle (in the original), we spend two parameters. Assuming the line segment AA_1 to be the representation of a perpendicular to the plane, we spend another two parameters. Finally, assuming the line segments AA_1 and AB

to be the representations of the line segments whose ratio of lengths is 1:2, we spend one more parameter. Thus, all the five parameters have been spent on the representation. Therefore, no new metric constructions are allowed to be carried out arbitrarily on this representation.

Further, let us note that the sought-for locus of points belongs to a certain plane passing through the point M, i.e. the midpoint of the line segment PQ perpendicular to this line segment. Thus, the problem is reduced to constructing the section of the prism by the plane passing through the point M perpendicular to the line segment PQ. To construct this cutting plane, we construct two straight lines, each of which passes through the point M perpendicular to PQ. Construct one of these straight lines, i.e. DE, which is parallel to the straight line AB in the plane ABQ. Since AB is perpendicular to PQ, we have: $DE \perp PQ$. To construct the second line, i.e. FK, which is perpendicular to PQ, first draw in the plane PCC_1 a straight line P_1C_1 parallel to PC. To compute the ratio $PK:PC$ in which the cutting plane divides the line segment PC, introduce an auxiliary parameter by setting, for instance, $AA_1 = a$. Then $AB = 2a$. If FK is perpendicular to PQ, then $\triangle PMK \sim \triangle PCQ$, and, therefore, $\frac{PK}{PQ} = \frac{PM}{PC}$, where $PC = a\sqrt{3}$, $PQ = \sqrt{PC^2 + QC^2} = \sqrt{(a\sqrt{3})^2 + \left(\frac{a}{2}\right)^2} = \frac{a\sqrt{13}}{2}$, and $PM = \frac{a\sqrt{13}}{4}$. We get: $PK = \frac{PQ \cdot PM}{PC} = \frac{13a}{8\sqrt{3}}$. Hence, $PK:PC = \frac{13a}{8\sqrt{3}}:a\sqrt{3}$, or $PK:PC = 13:24$. After having constructed a point K such that $PK:PC = 13:24$, we draw a straight line MK, which is perpendicular to the line segment PQ. By construction, FK is perpendicular to PQ. Thus, the cutting plane is specified by two straight lines, i.e. DE and FK.

Further, since $DE \parallel AB$, DE is parallel to the plane ABC, and this means that the cutting plane will intersect the plane ABC along a straight line parallel to DE. Bearing this in mind, we draw in the plane ABC, through the point K, a straight line GN parallel to DE. We then construct GE and ND, find the points R and T on the lines A_1C_1 and B_1C_1. Joining the points R and T, we get the quadrilateral $NGRT$ representing the section of the prism by the cutting plane perpendicular to the line segment PQ and passing through its midpoint. Thus, the quadrilateral $NGRT$ is the desired locus of points.

Remark. The construction of the straight line $FK \perp PQ$ could be carried out by means of an auxiliary drawing, as in the three previous examples. Namely, construct a rectangle similar to the original of the rectangle PP_1C_1C, that is, the rectangle $P'P_1'C_1'C'$ with the ratio of sides $P'P_1':P'C' =$

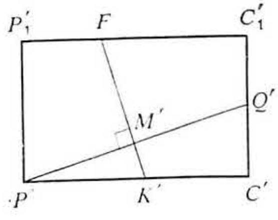


Fig. 122

$1:\sqrt{3}$ (Fig. 122). (If $AA_1 = a$, then in the equilateral triangle ABC, where $AB = 2a$, we shall have $PC = a\sqrt{3}$, whence $AA_1:PC = 1:\sqrt{3}$.)

In this rectangle we find Q', i.e. the midpoint of the line segment $C'C_1'$, and then M', i.e. the midpoint of the line segment $P'Q'$. Further, through the point M' we draw a straight line $M'K'$ perpendicular to $P'Q'$. This straight line divides the line segment $P'C'$ in a certain ratio $P'K':P'C'$. The line segment PC is divided in the same ratio by the point K. And further, we construct the cutting plane.

PROBLEMS TO BE SOLVED WITHOUT ASSISTANCE

I. Simplest Constructions in Space

563. Through a given straight line l draw a plane Q parallel to another given straight line m.

564. Construct two parallel planes P and Q so that the plane P passes through the straight line l, and the plane Q through the straight line m, where l and m are given skew lines.

565. Through a given point A draw a plane Q parallel to a given plane P.

566. Through a given point A draw a plane Q perpendicular to a given straight line l.

567. Through a given point A draw a straight line q perpendicular to a given plane P.

568. Through a given straight line l draw a plane Q perpendicular to a given plane P.

569. Draw a straight line q perpendicular to either of two given skew lines l and m and intersecting each of them.

II. Loci of Points

570. Find the locus of all points in space equidistant from two distinct points A and B.

571. Find the locus of all points in space equidistant from three given noncollinear points A, B, and C.

572. Find the locus of all points in space equidistant from two intersecting straight lines l_1 and l_2.

573. Find the locus of all points in space equidistant from four given points A, B, C, and D not belonging to one plane.

574. Find the locus of all points in space equidistant from four given points A, B, C, and D belonging to one plane.

575. Find the locus of all points in space equidistant from two given parallel straight lines l_1 and l_2.

576. Find the locus of all points in space equidistant from three given straight lines l_1, l_2, and l_3 passing through a common point and not belonging to one plane.

577. Find the locus of all points in space equidistant from three given straight lines l_1, l_2, and l_3 not lying in one plane if $l_1 \parallel l_2$ and $l_1 \parallel l_3$.

578. Find the locus of all points in space situated at a given distance from a given plane P.

579. Find the locus of all points in space equidistant from two parallel planes P_1 and P_2.

580. Find the locus of all points in space, the difference of the squares of distances from which to two distinct points A and B is a constant.

581. Find the locus of all points in space from which a given line segment AB is seen at a right angle.

582. Find the locus of all points in space, the sum of the squares of distances from which to two given points A and B is a constant.

583. Find the locus of all points in space, the ratio of the distances from which to two given points A and B is a constant.

III. Applications of Certain Loci of Points and Lines

584. Given two skew lines l_1 and l_2. Through a given point A belonging to neither l_1 nor l_2 draw a straight line q intersecting both given lines.

585. Through a given point A draw a straight line q perpendicular to two given skew lines l_1 and l_2.

586. Given three pairwise skew lines l_1, l_2, and l_3. Draw a straight line q intersecting the given lines at respective points L_1, L_2, and L_3 such that $L_1L_2 = L_2L_3$.

587. Through a given point A draw a straight line q parallel to a given plane P and intersecting a given straight line l.

588. Draw a straight line q intersecting two given straight lines l_1 and l_2 and parallel to a third given straight line l_3.

589. Draw a straight line q intersecting two given skew lines l and m, q being perpendicular to l and parallel to a given plane P.

590. Draw a straight line q intersecting two given straight lines l_1 and l_2, q being perpendicular to a third line l_3 and parallel to a given plane P.

591. Draw a straight line q belonging to a given plane P_1 parallel to another given plane P_2 and passing through the point of intersection of a given straight line l and the given plane P_1.

592. In a given plane P draw a straight line q perpendicular to a given straight line l (not lying in P) and passing through a given point A.

IV. Constructions on Representations

(1) *Constructing Plane Figures in Space*

593. Two altitudes are drawn in a triangle. Construct the centre of the circle circumscribed about this triangle.

594. The ratio of the legs of a right triangle is 3:4. Construct the centre of the circle inscribed in this triangle.

595. Given a right isosceles triangle. Construct a square lying in the plane of the triangle if the side of the square is represented by: (a) a leg of the given triangle; (b) the hypotenuse of the given triangle.

596. Given a regular hexagon, it is required to: (a) construct the apothem of the hexagon; (b) construct the bisector of one of its exterior angles; (c) drop a perpendicular from the centroid of the hexagon to one of its smaller diagonals.

597. The acute angle of a rhombus is equal to 45°. Construct the altitude of the rhombus.

(2) *Section of a Polyhedron by a Plane Parallel to Two Straight Lines*

598. Taken on the edges AA_1 and DD_1 of a parallelepiped $ABCDA_1B_1C_1D_1$ are points P and Q, respectively. Construct the section of the parallelepiped by the plane parallel to the straight lines B_1P and A_1Q and passing through the point K belonging to the edge: (a) CC_1; (b) DD_1; (c) A_1B_1; (d) AB.

599. Taken on the face AA_1B_1B of a triangular prism $ABCA_1B_1C_1$ is a point P and on the edge CC_1 a point Q. Construct the section of the prism by the plane parallel to the straight lines B_1P and A_1Q and passing through the point K belonging to the edge: (a) AA_1; (b) BB_1; (c) AC; (d) B_1C_1.

600. Taken on the edges AA_1 and CC_1 of a triangular prism $ABCA_1B_1C_1$ are points P and Q, respectively. Construct the section of the prism by the plane passing through the point K, i.e. the midpoint of the line segment PQ, parallel to the straight lines B_1P and AQ.

601. Taken in the plane of the base of a quadrangular prism $ABCDA_1B_1C_1D_1$ is a point P, and on the edge CC_1 a point Q. Construct the section of the prism

by the plane parallel to the straight lines B_1P and A_1Q and passing through the point K belonging to the edge: (a) DD_1; (b) A_1B_1; (c) AD; (d) B_1C_1.

602. Taken on the edges SA and SC of a triangular pyramid $SABC$ are points P and Q, respectively. Construct the section of the pyramid by the plane parallel to the straight lines BP and AQ and passing through the point K belonging to the edge: (a) SA; (b) SB; (c) AC; (d) BC.

603. Taken on the edges SA and SC of a triangular pyramid $SABC$ are points P and Q, respectively. Construct the section of the pyramid by the plane passing through the point K, i.e. the midpoint of the line segment PQ, parallel to the straight lines BP and AQ.

604. Taken on the edges SD and SC of a quadrangular pyramid $SABCD$ are points P and Q, respectively. Construct the section of the pyramid by the plane passing through the point K, i.e. the midpoint of the line segment PQ, parallel to the straight lines AP and DQ.

605. Taken on the edges SA and SC of a quadrangular pyramid $SABCD$ are points P and Q, respectively. Construct the section of the pyramid by the plane parallel to the straight lines BP and AQ and passing through the point K belonging to the edge: (a) SA; (b) AD; (c) DC; (d) BC.

606. Taken on the edges AA_1 and CC_1 of a quadrangular prism $ABCDA_1B_1C_1D_1$ are points P and Q, respectively. Construct the section of the prism equidistant from the plane passing through D_1P and parallel to DQ and from the plane passing through DQ and parallel to D_1P.

607. Drawn in a quadrangular prism $ABCDA_1B_1C_1D_1$ are the diagonals AC and A_1B of the faces $ABCD$ and ABB_1A_1, respectively. Construct the section of the prism by the plane equidistant from the plane passing through AC and parallel to A_1B, and from the plane passing through A_1B and parallel to AC.

(3) Constructing a Perpendicular to a Straight Line and a Perpendicular to a Plane

608. Taken on the edge SB of a regular tetrahedron $SABC$ is a point K, i.e. the midpoint of the edge. It is required to: (a) drop perpendiculars from the point K to the edges SA and SC; (b) find the point of intersection of the altitude SO of the tetrahedron and the plane passing through the perpendiculars dropped from the point K to the edges SA and SC of the tetrahedron.

609. In a rectangular parallelepiped $ABCDA_1B_1C_1D_1$ in which $AB:AD:AA_1 = 3:2:1$, construct the straight lines passing through the vertex B_1 and perpendicular to the lines BC_1 and BA_1.

610. A point P is taken on the edge CC_1 of a cube $ABCDA_1B_1C_1D_1$. Drop a perpendicular from the vertex A_1 to the line DP if: (a) $CP:PC_1 = 1:1$; (b) $CP:PC_1 = 1:2$; (c) $CP:PC_1 = 2:3$.

611. In a regular tetrahedron $SABC$, drop a perpendicular from the point K, i.e. the midpoint of the edge AC, to the plane of the face SBC.

612. In a regular quadrangular pyramid $SABCD$, the altitude SO is equal to the side of the base AB. From the vertex D drop a perpendicular to the plane of the face SBC.

613. The base of a prism is an equilateral triangle ABC. The faces ABB_1A_1 and ACC_1A_1 of the prism are rhombi each with an acute angle of 60°. Drop a perpendicular from the point P taken on the edge AA_1 to the diagonal BC_1 of the face BB_1C_1C if: (a) $AP:PA_1 = 1:1$; (b) $AP:PA_1 = 1:2$; (c) $AP:PA_1 = 2:3$.

614. In a cube $ABCDA_1B_1C_1D_1$, a plane is drawn through its vertices B, C_1, and D. A point P is taken on the edge A_1B_1. Drop a perpendicular from the point P to the plane BC_1D if: (a) $A_1P:PB_1 = 1:1$; (b) $A_1P:PB_1 = 1:2$; (c) $A_1P:PB_1 = 2:3$.

615. Taken on the edge A_1B_1 of a cube $ABCDA_1B_1C_1D_1$ is a point P and drawn through the points P, B, and C_1 is a plane. Drop a perpendicular from the vertex D to the plane PBC_1 if: (a) $A_1P:PB_1 = 1:1$; (b) $A_1P:PB_1 = 1:2$; (c) $A_1P:PB_1 = 2:3$.

616. In a rectangular parallelepiped $ABCDA_1B_1C_1D_1$ with the ratio of the edges $AB:AD:AA_1 = 1:2:1$, a plane is drawn through the vertices B, C_1, and D. A point P is taken on the edge A_1D_1. Drop a perpendicular from the point P to the plane BC_1D if: (a) $A_1P:PD_1 = 1:1$; (b) $A_1P:PD_1 = 1:2$; (c) $A_1P:PD_1 = 2:3$.

617. The base of a triangular pyramid $SABC$ is a right triangle ABC with the right angle at the vertex C and the ratio of the legs $AC:BC = 3:4$. The altitude of the pyramid is projected in the point C and is equal to the hypotenuse AB. Drop a perpendicular from the point C to the plane SAB.

(4) *Section of a Polyhedron by a Plane Passing Through a Given Point Perpendicular to a Given Line*

618. Taken on the edges DD_1 and CD of a cube $ABCDA_1B_1C_1D_1$ are points P and Q, respectively. Construct the section of the cube by the plane passing through the point P perpendicular to the straight line C_1Q if: (a) $DP:PD_1 = 1:1$, $DQ:QC = 1:1$; (b) $DP:PD_1 = 3:1$, $DQ:QC = 1:1$; (c) $DP:PD_1 = 3:1$, $DQ:QC = 3:1$; (d) $DP:PD_1 = 1:3$, $DQ:QC = 3:1$.

619. In a regular triangular pyramid $SABC$ the altitude is equal to the side of the base. Construct the section of the pyramid by the plane passing through the edge AB of the base perpendicular to the edge SC.

620. A point P is taken on the edge DD_1 of a rectangular parallelepiped $ABCDA_1B_1C_1D_1$ in which $AB:AD:AA_1 = 1:2:1$. Construct the section of the parallelepiped by the plane passing through the point K, i.e. the centroid of the face AA_1D_1D, perpendicular to the straight line C_1P if: (a) $DP:PD_1 = 1:1$; (b) $DP:PD_1 = 1:2$; (c) $DP:PD_1 = 2:3$.

621. Taken on the edges BB_1 and DD_1 of a right parallelepiped $ABCDA_1B_1C_1D_1$ in which $AB:AD:AA_1 = 1:1:2$ and the angle BAD is equal to 60° are points P and Q, respectively. Construct the section of the parallelepiped by the plane passing through the point Q perpendicular to the straight line AP if: (a) $BP:PB_1 = 1:1$, $DQ:QD_1 = 1:1$; (b) $BP:PB_1 = 1:1$, $DQ:QD_1 = 1:2$; (c) $BP:PB_1 = 1:2$, $DQ:QD_1 = 2:1$.

(5) *Constructing a Locus of Points Equidistant from Given Points*

622. In a regular triangular prism $ABCA_1B_1C_1$, $AA_1:AC = 1:2$. Construct the locus of points lying on the surface of the prism and equidistant from the points: (a) A and C_1; (b) P and C_1, where P is the midpoint of the edge AB; (c) Q and C_1, where Q is the midpoint of the edge AA_1.

623. Construct the locus of points belonging to the surface of a cube and equidistant from the following two points: (a) P and Q, belonging to the edges AB and CD, respectively; (b) B_1 and D, i.e. the end points of the diagonal B_1D of the cube; (c) M and K, i.e. the centroid of the face AA_1D_1D and the midpoint of the edge BB_1, respectively; (d) B_1 and O, i.e. the vertex of the cube and the centroid of the face $ABCD$.

624. Taken respectively on the edges A_1B_1, C_1C, and AD of a cube $ABCDA_1B_1C_1D_1$ are points P, Q, and R, i.e. the midpoints of these edges. Construct the points lying on the surface of the cube and equidistant from the following points: (a) D, D_1, and P; (b) D_1, P, and Q; (c) P, Q, and R.

625. Taken respectively on the edges AB and DD_1 of a cube $ABCDA_1B_1C_1D_1$ are points P and Q, i.e. the midpoints of these edges. Construct the point equidistant from the following four points: (a) A_1, C_1, P, and Q; (b) A, C_1, P, and Q; (c) D, C_1, P, and Q.

626. Taken respectively on the extensions of the edges AB and DD_1 of a cube $ABCDA_1B_1C_1D_1$ are points P and Q such that $AP:BP = 3:1$ and $D_1Q:DQ = 3:1$. Construct the point equidistant from the following four points: (a) A_1, C_1, P, and Q; (b) A, C_1, P, and Q; (c) D, C_1, P, and Q.

627. Taken respectively on the edges BB_1 and CC_1 of a regular triangular

prism $ABCA_1B_1C_1$ are points P and Q, i.e. the midpoints of these edges. The lateral faces of the prism are squares. Construct the locus of points equidistant from the following three points: (a) A, B, and Q; (b) A, B_1, and Q; (c) A, P, and Q.

628. Two congruent right isosceles triangles ABC and ABC_1 are arranged so that they have a common leg AB, and $CC_1 = BC$. On the straight lines BC, AB, and AC_1 construct the points equidistant from the points C and C_1.

629. Drawn in a cube $ABCDA_1B_1C_1D_1$ are the diagonal AD_1 and DC_1 of its lateral faces. Construct the locus of points equidistant from the straight lines AD_1 and DC_1.

630. Construct the locus of points equidistant from the straight lines containing the edges B_1C_1 and CD of the right prism $ABCDA_1B_1C_1D_1$ whose base is a square $ABCD$.

SEC. 10. SKEW LINES. ANGLE BETWEEN A STRAIGHT LINE AND A PLANE

Example 1. In a regular tetrahedron $SABC$, the line segment DO joins the midpoint D of the edge SA to the centroid O of the face ABC, and E is the midpoint of the edge SB. Find the angle between the straight lines DO and CE (Fig. 123).

Solution. Let the quadrilateral $SABC$ with its diagonals be the representation of the given tetrahedron. As it is not difficult to make sure, this representation is complete and metrically determined. Let D be the midpoint of the edge SA, E the midpoint of the edge SB, and O the centroid of the face ABC. Construct the straight lines DO and CE. (We do not spend any parameters on these constructions.) We then construct an angle equal to the angle between the straight lines DO and CE. To this end, find the point K, i.e. the intersection of the straight lines CO and AB, and join it to the point E. In the plane CEK, through the point O draw the straight line OF parallel to CE. Then, the angle DOF is equal to the angle between the straight lines DO and CE. We set for brevity $\angle DOF = x$. To determine x, we compute the sides of the triangle FOD and then apply the law of cosines.

To carry out computations, let us introduce an auxiliary parameter, setting $AB = a$. Then, since D is the midpoint of the edge SA, and SAO is a right triangle, we have: $OD = \frac{1}{2}SA = \frac{a}{2}$, and from the similarity of the triangles KFO and KEC: $\frac{OF}{CE} = \frac{OK}{CK} = \frac{1}{3}$, that is, $OF = \frac{1}{3}CE = \frac{a\sqrt{3}}{6}$. Further, in $\triangle DKF$, $DF^2 = DK^2 + KF^2 - 2DK \cdot KF \cdot \cos \angle DFK$. Since D, E, and K are the midpoints of the sides of the regular triangle SAB, $DK = \frac{1}{2} SB = \frac{a}{2}$ and $\angle DKF = 60°$. Besides, $KF = \frac{1}{3}KE = \frac{a}{6}$. Thus, we get: $DF^2 = \frac{7}{36}a^2$. Since $DF^2 < OD^2 + OF^2$, in the triangle

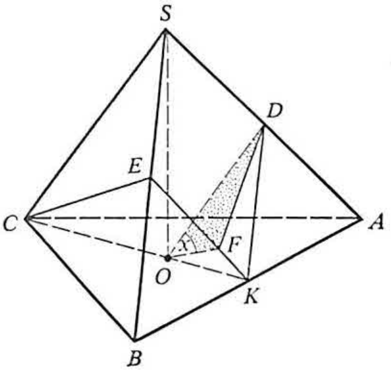


Fig. 123

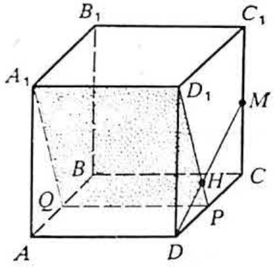


Fig. 124

DOF the angle DOF is acute (consequently, exactly this angle, and not the adjacent one, is the desired angle), and $DF^2 = OF^2 + OD^2 - 2OF \cdot OD \cdot \cos x$, whence $\cos x = \frac{5\sqrt{3}}{18}$.

Example 2. Given a cube $ABCDA_1B_1C_1D_1$ whose edge is equal to a. Taken on the edge DC is a point P, i.e. the midpoint of this edge. Find the distances between the following pairs of straight lines: (a) AA_1 and D_1P; (b) AD and D_1P; (c) BD and D_1P (Fig. 124).

Solution. Let the figure $ABCDA_1B_1C_1D_1$ be the representation of the given cube. This representation is complete and metrically determined, since all the five parameters have been spent for its construction. Since in solid geometry the representation of the original figure Φ_0 may be defined as any figure Φ similar to the parallel projection of the figure Φ_0 on some plane, we may regard the figure $ABCDA_1B_1C_1D_1$ as the representation of the cube with edge whose length is equal to a. On the obtained representation, we construct the point P, i.e. the midpoint of the line segment DC, and join the point P to D_1.

(a) Since A_1D_1 is an edge of the cube, it is clear that A_1D_1 is perpendicular to AA_1, and A_1D_1 is perpendicular to the plane CDD_1, that is, A_1D_1 is perpendicular to D_1P. But then $A_1D_1 = a$ is just the sought-for distance.

(b) Since the point P lies both in the plane A_1D_1P and in the plane ABC, these planes intersect along a straight line passing through the point P, say, along PQ. Since A_1D_1 is parallel to the plane ABC, PQ is parallel to A_1D_1. Let us also construct A_1Q. It is not difficult to show that AD is parallel to the plane A_1D_1P, and, thus, the distance between the straight lines AD and D_1P is equal to the distance from the straight line AD to the plane A_1D_1P.

Since the construction of the perpendicular from the point D to the straight line D_1P in the plane D_1DC must be carried out on

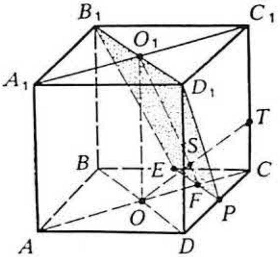


Fig. 125

a metrically determined representation, it is impossible to take arbitrarily a point H on the straight line D_1P and assume that DH is perpendicular to D_1P. We shall accomplish the construction in the following way. We construct the line segment DM, where M is the midpoint of the edge CC_1. Then from the congruence of the triangles DD_1P and DCM it follows that $\angle DPH = \angle CMD$, that is, $\angle HDP + \angle DPH = 90°$, and, consequently, in the triangle DPH the angle DHP is equal to 90°. Thus, $DH = D_1P$.

Let us show that DH is perpendicular to the plane A_1D_1P. Indeed, since AD is perpendicular to the plane DCC_1, we have: $DH \perp AD$ and $DH \perp PQ$. Thus, $DH \perp D_1P$ and $DH \perp PQ$, and, consequently, DH is perpendicular to the plane A_1D_1P. But then the length of the line segment DH is the required distance. Expressing the area of the triangle DD_1P in two ways, we get the equation with respect to DH: $DH \cdot D_1P = DP \cdot DD_1$, whence $DH = \frac{a\sqrt{5}}{5}$.

(c) In the plane ABC (Fig. 125), we construct the straight line EK passing through the point P and through E, i.e. the midpoint of the edge BC. Then BD is parallel to EP. We then construct the plane B_1EP. Since BD is parallel to EP, BD is parallel to the plane B_1EP, and, thus, the required distance is equal to the distance from the straight line BD to the plane B_1EP. To find this distance, we draw a perpendicular from the point O to the plane B_1EP. The foot of this perpendicular, i.e. the point S, cannot be taken arbitrarily, since the representation is metrically determined. The point S is constructed as follows. We construct the point F, i.e. the point of intersection of the straight lines OC and EP, then the point O_1, i.e. the point of intersection of the straight lines A_1C_1 and B_1D_1, and then join the points O_1 and F. We then take on the edge CC_1 a point T such that the equality $\frac{CT}{OC} = \frac{OF}{OO_1}$ is fulfilled. Since $OC = \frac{a\sqrt{2}}{2}$, $OF = \frac{a\sqrt{2}}{4}$, and $OO_1 = a$, we have: $CT = \frac{a}{4}$. Let us construct the line segment OT and the point S, i.e. the point of intersection of OT and O_1F. Since, by construction, the right triangles OO_1F and OCT are similar, we have: $\angle O_1FO = \angle OTC = \angle COT$, then $\angle O_1FO + \angle COT = 90°$, and, consequently, $OT \perp O_1F$. Let us now prove that OT is perpendicular to the plane B_1EP. Since B_1D_1 is perpendicular to the plane AA_1C, and OT lies in the plane AA_1C, OT is perpendicular to B_1D_1. Thus, $OT \perp B_1D_1$ and $OT \perp O_1F$. Hence, OT is perpendicular to the plane B_1EP. And then the length of the line segment OS is the required distance. Expressing the area of the triangle O_1OF in two ways, we get the following equation with respect to OS: $OS \cdot O_1F = OF \cdot OO_1$, whence, since $OF = \frac{a\sqrt{2}}{4}$, $OO_1 = a$, and $O_1F = \frac{3a\sqrt{2}}{4}$, we find that $OS = \frac{a}{3}$.

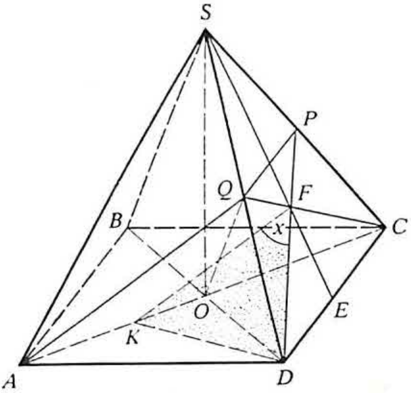


Fig. 126

Example 3. The lateral surface area of a regular quadrangular pyramid $SABCD$ is twice the area of its base. Drawn in the faces SAD and SDC are the respective medians AQ and DP. Find the angle between them (Fig. 126).

Solution. First Method. Let the figure $SABCD$ be the representation of the given pyramid. This representation is complete with all the five parameters spent on it, that is, it is also metrically determined. Indeed, assuming the parallelogram $ABCD$ as the representation of a square, we spend two parameters. Assuming SO as the representation of a line segment perpendicular to the plane $ABCD$, we also spend two parameters. Finally, assuming that $S_{\text{lat}} = 2S_{ABCD}$, we actually assume that the altitude of a lateral face of the pyramid is equal to the side of its base. (Setting SE, i.e. the height of the lateral face, to be equal to h, and the side DC of the base of the pyramid to a, we find from the equality $S_{\text{lat}} = 2S_{ABCD}$ that $4 \cdot \frac{1}{2} ah = 2a^2$, whence it follows that $h = a$.) Thus, there are no parameters for additional constructions of the metric character. Note that the chosen method of solution will not involve constructions requiring a further expenditure of parameters.

To determine the desired angle, we include this angle in some triangle. This can be done, for instance, in the following manner: join the points Q and C (CQ the median of the triangle SCD) and denote the point of intersection of the straight lines DP and CQ by F. In the triangle AQC, through the point F draw the straight line KF parallel to AQ, and, finally, join the point K to the point D. Since, by construction, KF is parallel to AQ, the angle between the straight lines KF and DF is equal to the required angle. For the sake of brevity, we set $\angle KFD = x$. To determine x, let us compute the sides of the triangle DFK.

Introduce an auxiliary parameter, setting $DC = a$. Then, as it was noted when counting the parameters of the representation, $SE = a$. But SE is the median of the triangle SDC, that is, $FE =$

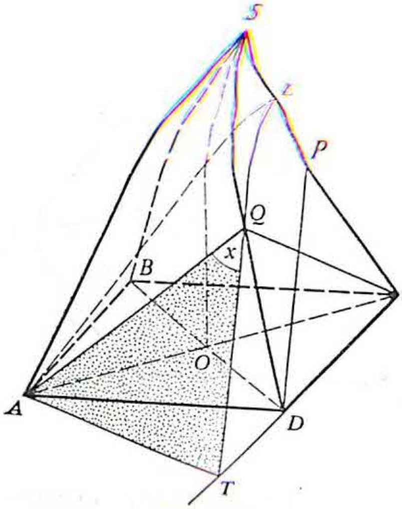


Fig. 127

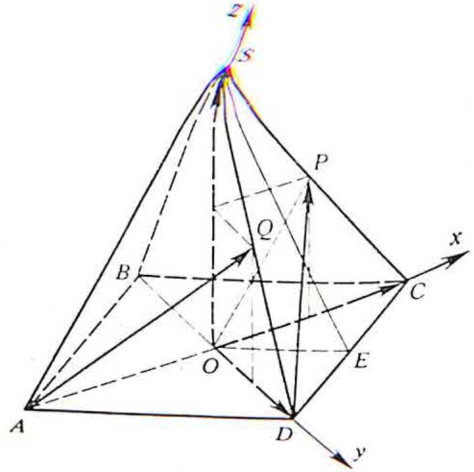


Fig. 128

$\frac{a}{3}$. Then we find from the triangle DFE, where $DE=\frac{a}{2}$, that $DF=\frac{a\sqrt{13}}{6}$. Further, since the triangle CFK is similar to the triangle CQA, we have: $\frac{KF}{AQ}=\frac{CF}{CQ}$. But $\frac{CF}{CQ}=\frac{2}{3}$, and, consequently, $KF=\frac{2}{3}AQ$. But $AQ=DP$, and, hence, $KF=\frac{2}{3}DP$. Bearing in mind that $DF=\frac{2}{3}DP$, we arrive at the conclusion that $KF=DF$, i.e. $KF=\frac{a\sqrt{13}}{6}$. The third side of the triangle DFK is found from the triangle ADK. We get: $DK^2=AK^2+AD^2-2AK\cdot AD\cdot\cos\angle DAK$, where $AK=\frac{1}{3}AC=\frac{a\sqrt{2}}{3}$, $AD=a$, and $\angle DAK=45^\circ$. Then $DK^2=\frac{5a^2}{9}$. Thus, in the triangle DFK all the three sides are known. Applying the law of cosines to this triangle, we get: $DK^2=KF^2+FD^2-2KF\cdot FD\cdot\cos x$, or $\frac{5a^2}{9}=\frac{13a^2}{36}+\frac{13a^2}{36}-2\frac{a\sqrt{13}}{6}\cdot\frac{a\sqrt{13}}{6}\cos x$, whence $\cos x=\frac{3}{13}$, i.e. $x=\arccos\frac{3}{13}$.

Remark. Let us indicate another way of including the required angle in a triangle. In the plane SDC, through the point Q draw the straight line QT parallel to DP and join the point T to the point A (Fig. 127). It is clear that the angle TQA is congruent to the desired one. It can be found from the triangle TQA. We may proceed in another way, namely: from the triangle AQL we find the angle AQL, which supplements the desired angle to 180°, and then determine the desired angle itself.

Second Method. Let us introduce in space the rectangular Cartesian basis $Oijk$, the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ of which are the unit vectors of x-, y-, and z-axes (Fig. 128). Decomposing then the vectors $\overrightarrow{DP}$ and $\overrightarrow{AQ}$ along the vectors of this basis, we find $\cos\angle(\overrightarrow{AQ}, \overrightarrow{DP})$. We then introduce an auxiliary parameter, setting $DC = a$. Then $OD = OC = \frac{a\sqrt{2}}{2}$, and we find from the triangle SOE, where $SE = a$ and $OE = \frac{a}{2}$, that $SO = \frac{a\sqrt{3}}{2}$. Now express the vectors $\overrightarrow{DP}$ and $\overrightarrow{AQ}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. We have: $\overrightarrow{DP} = \overrightarrow{OP} - \overrightarrow{OD}$, where $\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OC} + \frac{1}{2}\overrightarrow{OS} = \frac{a\sqrt{2}}{4}\mathbf{i} + \frac{a\sqrt{3}}{4}\mathbf{k}$, $\overrightarrow{OD} = \frac{a\sqrt{2}}{2}\mathbf{j}$. Then $\overrightarrow{DP} = \frac{a\sqrt{2}}{4}\mathbf{i} - \frac{a\sqrt{2}}{2}\mathbf{j} + \frac{a\sqrt{3}}{4}\mathbf{k}$. Further, $\overrightarrow{AQ} = \overrightarrow{OQ} - \overrightarrow{OA}$, where $\overrightarrow{OQ} = \frac{1}{2}\overrightarrow{OD} + \frac{1}{2}\overrightarrow{OS} = \frac{a\sqrt{2}}{4}\mathbf{j} + \frac{a\sqrt{3}}{4}\mathbf{k}$ and $\overrightarrow{OA} = -\frac{a\sqrt{2}}{2}\mathbf{i}$. Thus $\overrightarrow{AQ} = \frac{a\sqrt{2}}{2}\mathbf{i} + \frac{a\sqrt{2}}{4}\mathbf{j} + \frac{a\sqrt{3}}{4}\mathbf{k}$. Hence

$$\begin{aligned}\cos\angle(\overrightarrow{AQ}, \overrightarrow{DP}) &= \frac{\overrightarrow{AQ}\cdot\overrightarrow{DP}}{|\overrightarrow{AQ}|\cdot|\overrightarrow{DP}|}\\ &= \left(\frac{a\sqrt{2}}{4}\cdot\frac{a\sqrt{2}}{4} - \frac{a\sqrt{2}}{2}\cdot\frac{a\sqrt{2}}{4} + \frac{a\sqrt{3}}{4}\cdot\frac{a\sqrt{3}}{4}\right)\\ &\times\left[\sqrt{\left(\frac{a\sqrt{2}}{4}\right)^2 + \left(\frac{a\sqrt{2}}{2}\right)^2 + \left(\frac{a\sqrt{3}}{4}\right)^2}\right.\\ &\times\left.\sqrt{\left(\frac{a\sqrt{2}}{2}\right)^2 + \left(\frac{a\sqrt{2}}{4}\right)^2 + \left(\frac{a\sqrt{3}}{4}\right)^2}\right]^{-1} = \frac{3}{13},\end{aligned}$$

that is, $\angle(\overrightarrow{AQ}, \overrightarrow{DP}) = \arccos\frac{3}{13}$.

Example 4. The base of a quadrangular pyramid $SABCD$ is a rectangle with the ratio of the sides $AB:AD = 1:3$. Each lateral edge makes an angle of 60° with the plane of the base. Find the angle formed by the straight line DP and the plane SCD if P is the midpoint of the edge SB (Fig. 129).

Solution. Let the figure $SABCD$ be the representation of the given pyramid. This representation is complete. Let us compute its parametric number. An arbitrary parallelogram $ABCD$ is assumed to be the representation of the rectangle (that is, one parameter is spent) with the known ratio of sides (consequently, one more parameter is spent). Let us drop the perpendicular SO from the vertex S to the plane of the base (that is, we shall spend two more parameters

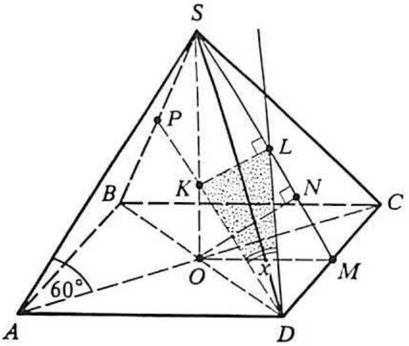


Fig. 129

on the representation) and join the point O to the vertex A. The angle SAO is made by the lateral edge SA and the plane of the base. We shall regard the angle SAO as the representation of the angle equal to 60° in the original (spending thereby one more parameter). Thus, we have spent all the five parameters on the representation, and this means that the representation is metrically determined, that is, any further metric constructions are not allowed to be performed arbitrarily.

Further, we join the point O to the vertex C of the base. Then, $\angle SAO = \angle SCO = 60^\circ$, and the right triangles SAO and SCO, which have a common leg SO and an equal acute angle, are congruent. And this means that $OA = OC$, that is, O is the midpoint of the diagonal AC, and, hence, the point of intersection of the diagonals of the rectangle $ABCD$. From the congruence of the triangles SAO and SCO it also follows that $SA = SC$, that is, SAC is an isosceles triangle. Since $\angle SAO = \angle SCO = 60^\circ$, we have: $\angle ASC = 60^\circ$, and, therefore, the triangle SAC is equilateral. Analogously, the triangle SBD is also equilateral.

Now, let us pass over to constructing the desired angle. Since the angle between a straight line and a plane is equal to the angle formed by the line and its orthogonal projection on this plane, we drop a perpendicular from some point on the straight line DP to the plane SCD. To this end, we first draw the slant height SM of the face SCD (M the midpoint of the edge CD) and then join the point M to the point O. Then, since SM is perpendicular to CD, and OM is the projection of SM on the plane $ABCD$, OM is also perpendicular to CD. Thus, $CD \perp SM$ and $CD \perp OM$, and this means that the straight line CD is perpendicular to any straight line lying in the plane SOM. Therefore, if we drop the perpendicular KL from the point K to the slant height SM, then KL will also be perpendicular to the plane SCD, and, consequently, $\angle KDL$ will be the required angle. Let us set $\angle KDL = x$. As it was noted, new constructions of metric character are forbidden to be performed arbitrarily on the representation of the given pyramid, that is, taking an arbitrary point L on the slant height SM, it is impossible to say that the line segment KL will be assumed as the representation of the perpendicular to SM. To construct KL perpendicular to SM, let us first construct ON perpendicular to SM. This can be done if we compute in what ratio the point N divides the hypotenuse SM in the triangle SOM.

Setting $AB = a$, we get in succession: $BC = 3a$, $OM = \frac{3a}{2}$, $AC = a\sqrt{10}$, $SC = a\sqrt{10}$, $OC = \frac{a\sqrt{10}}{2}$, $SO = \frac{a\sqrt{30}}{2}$, $SM = \frac{a\sqrt{39}}{2}$, and $MN = \frac{3a\sqrt{39}}{2}$, that is, $MN:SM = 3:13$. Constructing a point N such that $MN:SM = 3:13$, we then draw $KL \parallel ON$. Now, we get from the similarity of the triangles SKL and SOM: $\frac{KL}{OM} = \frac{SK}{SM}$, where $OM = \frac{3a}{2}$, $SM = \frac{a\sqrt{39}}{26}$, and since K is the point of intersection of the medians SO and DP of the triangle SBD, we have: $SK = \frac{2}{3} SO = \frac{a\sqrt{30}}{3}$. Thus, $KL = \frac{OM \cdot SK}{SM} = \frac{a\sqrt{30}}{\sqrt{39}}$. Finally, we find from the right triangle DKL that $\sin x = \frac{KL}{DK}$. But $DK = SK = \frac{a\sqrt{30}}{3}$. Thus, $\sin x = \frac{\sqrt{39}}{13}$, that is, $x = \arcsin \frac{\sqrt{39}}{13}$.

Example 5. The base of a quadrangular pyramid $SABCD$ with vertex at the point S is a square, and its altitude is projected in the point A. The area of the face SBC is twice the area of the face SAB. Find the angle between the altitude AF of the face SAB and the diagonal plane SAC (Fig. 130).

Solution. Let the figure $SABCD$ be the representation of the given pyramid. This representation is complete. Let us determine the number of the parameters spent on this representation. Assuming the parallelogram $ABCD$ to be the representation of the square, we spend two parameters. Assuming the line segment SA to be the representation of the perpendicular to the plane of the base, we spend two more parameters. Finally, assuming the area of the face SBC to be twice the area of the face SAB, that is, $\frac{1}{2} SB \cdot BC = 2 \cdot \frac{1}{2} \cdot SA \cdot AB$, or $SB = 2SA$, we spend one more parameter. Thus, we have spent all the five parameters on the representation, and, consequently, it is metrically determined. This means that now it is impossible to take arbitrarily a point F on the edge

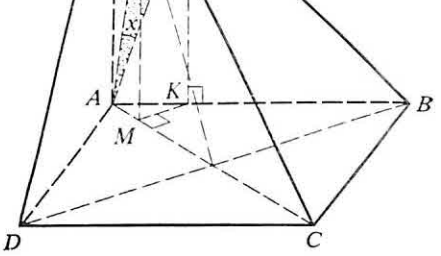


Fig. 130

SB and assume that the line segment AF is the representation of the altitude of the face SAB. However, to determine the desired angle, the altitude AF should be constructed necessarily (and then it will also be necessary to construct the representation of its projection on the plane SAC).

To carry out necessary constructions and further calculations, let us introduce an auxiliary parameter, setting, for instance, $AB = a$. Then, since in the right triangle SAB, $SA = \frac{1}{2} SB$, $\angle SBA = 30°$, and, hence, if $AF \perp SB$, then in the right triangle ABF, $AF = \frac{1}{2} AB$, that is, $AF = \frac{a}{2}$. Then $SA = \frac{a\sqrt{3}}{3}$, $SB = \frac{2a\sqrt{3}}{3}$, and $SF = \frac{a\sqrt{3}}{6}$. Thus, $SF:SB = \frac{a\sqrt{3}}{6} : \frac{2a\sqrt{3}}{3}$, or $SF:SB = 1:4$, whence it is clear how the altitude AF is constructed.

Remark. The altitude AF could also be constructed by means of an auxiliary drawing of a triangle similar to the original of the triangle SAB. Such a method of construction was considered in detail in Examples 7, 8, and 9 of the preceding section.

Then it is required to construct the representation of the projection of the altitude AF on the plane SAC. This construction can be carried out in the following way. In the plane SAB, through the point F draw a straight line parallel to the straight line SA. Denote the intersection of this line and the straight line AB by K. In the plane ABC, through the point K draw a straight line parallel to the straight line BD. Denote the intersection of this line and straight line AC by M. In the plane SAC, through the point M draw a straight line m along which the planes SAC and FKM intersect. Since both planes, SAC and FKM, are perpendicular to the plane ABC, it is clear that the line of their intersection, i.e. m, will also be perpendicular to the plane ABC, that is, $m \parallel SA$ and $m \parallel FK$. Denote the intersection of the straight line m and the plane SBD by L. Then join the point F to the point L. The line segment FL will represent the perpendicular to the plane SAC. (Indeed, $FL \parallel MK$, but MK is perpendicular to the plane SAC, since $MK \perp AC$ and $MK \perp SA$.) But then AL is the projection of AF on the plane SAC, and, consequently, $\angle FAL$ is the required angle. Let us set for brevity $\angle FAL = x$. To determine x, it is possible to find FL and then $\sin x = \frac{FL}{AF}$. The altitude AF was found above: $AF = \frac{a}{2}$. To compute FL, note that, by construction, the quadrilateral $MKFL$ is a rectangle, that is, $FL = KM$. Further, KM can be found from the similarity of the triangles AKM and ABC, whence $\frac{KM}{BC} = \frac{AK}{AC}$, where $BC = a$, $AC = a\sqrt{2}$, and $AK = \frac{a}{4}$ (since in the triangle SAB, $SF{:}SB = 1{:}4$, and, hence, $AK{:}AB = 1{:}4$). Thus, $KM = \frac{a}{4\sqrt{2}}$. We now find that $\sin x = \frac{a}{4\sqrt{2}} \div \frac{a}{2} = \frac{\sqrt{2}}{4}$, and, thus, $\angle FAL = \arcsin \frac{\sqrt{2}}{4}$.

PROBLEMS TO BE SOLVED WITHOUT ASSISTANCE

631. An inclined line AB forms with a plane P an angle of 45° equal to the angle between the projection of this line and the line AC lying in the plane P. Find the angle BAC.

632. In a rectangular triangular pyramid, the altitude is equal to the side of the base. Find the angle formed by the lateral edge and the plane of the base.

633. A straight line AB lies in the plane P. Drawn through the point B on one side of this line are straight lines BC and BD perpendicular to the line AB and forming angles equal to 50° and 15° with the plane P. Find the angle CBD.

634. The diagonal of a regular quadrangular prism forms an angle of 45° with the plane of the base. Find the angle formed by this diagonal and the diagonal of a lateral face (the diagonals intersect each other).

635. The vertex angle of the axial section of a cone is equal to 2α. A plane is passed through this vertex so that the altitude of the cone makes an angle β with it. Find the angle between the elements along which this plane intersects the surface of the cone.

636. Drawn in a cube $ABCDA_1B_1C_1D_1$ is a cutting plane PQC_1B_1, where P and Q are the midpoints of the edges AA_1 and DD_1, respectively. Find the angle between the straight line CQ and the cutting plane.

637. In a regular tetrahedron $SABC$ find the angle between the edge AB and the plane of the face SAC.

638. Drawn in a regular quadrangular pyramid $SABCD$ through the diagonal AC is a cutting plane perpendicular to the face SAD. What is the size of the angle formed by the edge SD and the cutting plane?

639. In a regular quadrangular pyramid $SABCD$, the ratio of the altitude SO to the side AB of the base is 2:3. Taken on the diagonal AC is a point P such that $AP = PO$. Find the angle made by the straight line SP and the plane of the face SAD.

640. The base of a quadrangular pyramid $SABCD$ is a square $ABCD$. In the lateral face SAB, which is a regular triangle perpendicular to the plane of the base, the median AK is drawn. The point K is joined to the vertex C. Find the angles formed by the sides AK and CK of the triangle ACK and the plane of the base.

641. The base of a pyramid $SABCD$ is a rectangle with sides $AD = a$ and $AB = b$. The altitude of the pyramid is projected in the point O, i.e. the point of intersection of the diagonals of the base. Each lateral edge forms an angle of 30° with the plane of the base. Find the angle made by the straight line DP and the plane SAC if P is the midpoint of the altitude SO.

642. The base of a right parallelepiped $ABCDA_1B_1C_1D_1$ is a parallelogram with an acute angle at the vertex A equal to 60°. Find the angle between the diagonal B_1D of the parallelepiped and the lateral face CC_1D_1D if $AB{:}AD{:}AA_1 = 2{:}1{:}3$.

643. In a cube $ABCDA_1B_1C_1D_1$, P is the midpoint of the edge AB. Find the angle between the straight line C_1P and the diagonal section AA_1C_1C of the cube.

644. Drawn in a cube $ABCDA_1B_1C_1D_1$ are a cutting plane BDC_1 and a diagonal CD_1 of the lateral face CC_1D_1D. Find the angle between the diagonal CD_1 and the cutting plane.

645. The base of a pyramid $SABCD$ is a rectangle with the ratio of the sides $AB:BC = 1:2$. Each lateral edge is inclined at an angle of 60° to the plane of the base. Find the angles formed by the straight lines DP and DQ and the diagonal plane SAC if P and Q are the midpoints of the edges SA and SC, respectively.

646. In a regular quadrangular pyramid, each lateral edge forms an angle of 45° with the plane of the base. A straight line is drawn through one of the vertices of the base and the midpoint of the altitude of the pyramid. Find the angle between the straight line and the lateral faces of the pyramid.

647. Drawn in a cube $ABCDA_1B_1C_1D_1$ is a plane C_1DB and a straight line DP, where P is the midpoint of the edge BB_1. Find the angle between the straight line DP and the cutting plane.

648. Drawn in a cube $ABCDA_1B_1C_1D_1$ is a cutting plane BDP, where P is the midpoint of the edge CC_1. Find the angle between the straight line A_1Q and the cutting plane if Q is the midpoint of the edge DD_1.

649. The base of a pyramid $SABC$ is an isosceles triangle with a right angle C. Each lateral edge makes an angle of 45° with the plane of the base. Find the angle between the median AM of the face SAB and the plane of the face SBC.

650. A straight line touches a cone and makes an acute angle α with the generatrix of the cone at the point of tangency, and an angle β with the plane of the cone's base. Find the angle between the generatrix and the plane of the base.

651. A circle of radius R and an equilateral triangle, $R\sqrt{3}$ on a side, lie in mutually perpendicular planes. The line segment joining the centre of the circle to the centroid of the triangle makes angles equal to 30° (each) with the given planes, and one of the sides of the triangle belongs to the plane of the circle. Find the length of the part of the side of the triangle which lies inside the circle.

652. In a regular quadrangular prism $ABCDA_1B_1C_1D_1$, the diagonals B_1D and BD_1 are mutually perpendicular. Find the angle between the diagonals B_1D and A_1C.

653. A square with a diagonal drawn in it is bent to form the lateral surface of a regular quadrangular prism, and, thus, the diagonal of the square became a nonplanar polygonal line. Find the angles between the segments of this polygonal line.

654. In a regular tetrahedron $SABC$, D is the midpoint of the edge SA, and E is the midpoint of the altitude. Find the angle between the straight lines OD and CE.

655. Prove that in a regular triangular pyramid, the pairs of nonintersecting edges are mutually perpendicular.

656. Prove that if the opposite edges of a triangular pyramid are pairwise perpendicular, then all the altitudes of the pyramid are concurrent (that is, intersect at a common point).

657. In a regular tetrahedron $SABC$, AD is the median of the triangle ABC, and E is the midpoint of the edge SB. Find the angle between the straight lines AD and CE.

658. In a tetrahedron $SABC$, SA is perpendicular to the plane ABC, $AB \perp AC$, and $SA = AB = AC$. Find the angle between the straight lines OD and CE if D is the midpoint of the edge SA, E the midpoint of the edge SB, and O is the centroid of the base ABC of the tetrahedron.

659. The base of a right parallelepiped $ABCDA_1B_1C_1D_1$ is a parallelogram $ABCD$ with an acute angle DAB equal to γ. The diagonals AB_1 and BC_1 of the lateral faces form angles equal to α and β, respectively, with the plane of the base. Find the angle between these diagonals.

660. The angle between the skew lines a and b is equal to 60°. The distance from their common perpendicular to the point A lying on the line a is equal to the distance from this perpendicular to the point B on the line b, and is equal to the distance between the lines a and b. Find the angle between the common perpendicular and the straight line AB.

661. In a cube $ABCDA_1B_1C_1D_1$, taken respectively on the edges AA_1 and DD_1 are points P and Q, i.e. the midpoints of these edges. Find the angle between the rays DP and QC_1.

662. In a regular tetrahedron $SABC$ find the angle between the median AM of the face SAB and the ray SC.

663. In a regular triangular pyramid $SABC$, all the plane angles at the vertex S are right. Find the angle between the rays CP and SD if P and D are the midpoints of the edges SA and BC, respectively.

664. In a quadrangular pyramid $SABCD$, the base $ABCD$ is a parallelogram with the ratio of the sides $AB:BC = 1:2$ and an acute angle equal to 60°. The face SAB of the pyramid is perpendicular to the plane of the base and is a regular triangle. Find the angle between the median DM of the face SAD and the slant height SK of the face SAB.

665. In a quadrangular pyramid $SABCD$, all the lateral edges are inclined at an angle of 60° to the plane of the base. A straight line is drawn through the point F, which divides the diagonal AC so that $AF:FC = 1:3$, and P, i.e. the midpoint of the edge SC. Find the angle made by the ray FP and the diagonal AC.

666. In a regular triangular prism $ABCA_1B_1C_1$ whose lateral faces are squares, the diagonals BA_1, AC_1, and CB_1 are drawn. Find the angles between the rays AC_1 and BA_1, AC_1 and CB_1, BA_1 and CB_1.

667. In a rectangle $ABCD$, $AB:BC = 2:1$. Taken respectively on the sides AB and CD are points P and Q, i.e. the midpoints of these sides; the rectangle is bent along the straight line PQ so that the angle between the rays PB and PA is equal to 60°. Find the angle between the rays DB and AQ.

668. In a regular quadrangular pyramid $SABCD$, the angle between each lateral edge and the plane of the base is equal to 45°. Find the angle between the rays BP and OL, where B is one of the vertices of the base of the pyramid, P the midpoint of the altitude of the pyramid, O the centroid of the base, and L the midpoint of the edge SC.

669. A square plate $ABCD$ is bent along the diagonal AC so that the plane ABC becomes perpendicular to the plane ACD. Chosen on the diagonal AC is a point K such that $CK:KA = 1:3$. Find the angle between the rays KB and CD.

670. In a triangular pyramid $SABC$, the plane angles ASB and CSB at the vertex S are equal to 90° (each), and the angle ASC to 45°. Find the angle between the rays KC and SD if K and D are the respective midpoints of the edges SA and BC, and $SA = SB = SC$.

671. In a triangular pyramid $SABC$, the base ABC is a regular triangle, the faces SAB and SBC are perpendicular to the plane of the base, and the edge SB is equal to the side of the base. Find the angle between the rays SO and BD, where O is the centroid of the triangle ABC, and D the midpoint of the edge SC.

672. The diagonals ad_1 and bc_1 of the faces of a rectangular parallelepiped $abcda_1b_1c_1d_1$ form angles α and β with the plane of the base. Find the angle between these diagonals.

673. In a regular quadrangular pyramid $SABCD$, the side of the base $AB =$ 6 cm, the altitude being equal to 4 cm. Find the distance from the vertex A to the plane of the face SCD.

674. The line segment AB whose length is equal to a is parallel to the plane P. Drawn through the points A and B are straight lines perpendicular to the line segment AB and forming angles equal to α and β, respectively, with the plane P. The distance between the points at which the drawn lines inter-

sect the plane P is equal to b. Find the distance between the line segment AB and the plane P.

675. A line segment, whose length is equal to a and whose end points lie on two mutually perpendicular planes, makes an angle equal to 30° with one of these planes, and an angle equal to 45° with the other. Find the distance between the projections of the end points of the given segment on the line of intersection of the planes.

676. Situated in the plane P is an equilateral triangle ABC in which $AB = a$. On the perpendicular to the plane P, the former passing through the point A, a line segment AK is laid off such that $AK = a$. Find the distance between the straight lines AB and CK.

677. Drawn through the upper end point of the generatrix of a cylinder at an angle α is a tangent to the cylinder. Find the distances from the centres of the bases of the cylinder to this tangent if the altitude of the cylinder and the radius of its base are equal to h and R, respectively.

678. Erected at the points A and B belonging to the plane P are the perpendiculars AC and BD to the plane P on one side of it. Prove that the straight lines BC and AD intersect and find the distance from the point of their intersection to the plane P if it is known that $AC = a$ and $BD = b$.

679. The altitude of a regular quadrangular prism is equal to h. Find the distance from the side of the base whose length is a to a nonintersecting diagonal of the prism.

680. The base of a right prism $ABCDA_1B_1C_1D_1$ is a rhombus with side equal to a and an acute angle φ. Find the distance from the vertex B_1 of the upper base to the diagonal A_1D of the lateral face if the lateral edge of the prism is equal to h.

681. Find the distance between a diagonal of a cube and a nonintersecting edge if the edge of the cube is equal to a.

682. Find the distance between the diagonals ad_1 and bc_1 of the faces of a cube $abcda_1b_1c_1d_1$ if its edge is equal to a.

683. In an equilateral cylinder, the radius of whose base is equal to R, a point on the circumference of the upper base is joined to a point on the circumference of the lower base. The line segment thus obtained forms an angle equal to α with the plane of the base. Find the distance between this line segment and the axis of symmetry of the cylinder.

684. Drawn between two parallel planes are a perpendicular and an inclined line making an angle α with each of the planes. Find the distance between the midpoints of the segments of the inclined line and the perpendicular enclosed between the given planes if the length of the segment of the perpendicular is equal to $2a$, and the distance between the end points of the inclined line and the perpendicular in each plane is equal to b.

685. In a triangular pyramid, the sum of three plane angles at each of the vertices of the base is equal to 180°. Find the distance between the skew edges of the pyramid if the sides of the base are equal to 4 cm, 5 cm, and 6 cm.

686. Prove that if a straight line a forms equal angles with three nonparallel straight lines lying in one plane, then the straight line a is perpendicular to this plane.

687. The plane angles at one of the vertices of a triangular pyramid are equal to 90° each. Prove that the altitude of the pyramid drawn from this vertex passes through the point of intersection of the altitudes of the opposite face.

688. Prove that the altitude SO of a triangular pyramid $SABC$ intersects the altitude AD of the base if and only if SA is perpendicular to BC.

689. Prove that if one of the altitudes of a triangular pyramid passes through the point of intersection of the altitudes of the opposite face, then the other altitudes of this pyramid possess the same property.

SEC. 11. DIHEDRAL AND POLYHEDRAL ANGLES

Example 1. One of the plane angles of a trihedral angle is equal to 60°, and each of the two others contains 45°. Find the dihedral angle opposite to the plane angle equal to 60°.

Solution. Let the figure $SMNL$ (Fig. 131) be the representation of the given trihedral angle. This representation is complete. Let us find its parametric number. Assuming the angle MSN to be the representation of the angle of 60°, we spend one parameter. Assuming the angles MSL and NSL to be the representations of the original angles each containing 45°, we spend two more parameters. Thus, we have spent only three parameters, that is, when carrying out new constructions of metric character, which may be required in determining the desired dihedral angle, we may spend another two parameters.

Thus, $\angle MSN = 60°$ and $\angle MSL = \angle NSL = 45°$. Find $\angle SL$, i.e. the dihedral angle at the edge SL. To this end, we construct and then find its plane angle. The construction is carried out in the following way. On the ray SL we choose an arbitrary point A, and in the plane MSL construct a straight line AB, which will be assumed the representation of the perpendicular to straight line SL (we spend one parameter). Analogously, we shall assume AC to be the representation of the perpendicular to the ray SL (here, we spend one more parameter). Now, all the five parameters have been spent on the representation. Thus, BAC is the desired plane angle of the dihedral angle SL.

Let us perform all necessary calculations, introducing an auxiliary parameter. We set, for example, $SA = a$. Then we find from the right triangles SAB and SAC that $AB = a$, $AC = a$, and $SB = SC = a\sqrt{2}$. Since in the triangle SBC, $BC = SB = SC$, we have: $BC = a\sqrt{2}$.

Let us make sure that the equality $AB^2 + AC^2 = BC^2$ is fulfilled. Indeed, $AB^2 + AC^2 = 2a^2$ and $BC^2 = (a\sqrt{2})^2 = 2a^2$. Thus, in the triangle ABC (by the converse of the Pythagorean theorem) $\angle BAC = 90°$. Hence, the angle SL is also equal to 90°.

Remark. The above problem could also be solved without introducing an auxiliary parameter. Indeed, as it is not difficult to note, $\triangle ABC = \triangle SAB$, therefore $\angle BAC = \angle SAB = 90°$.

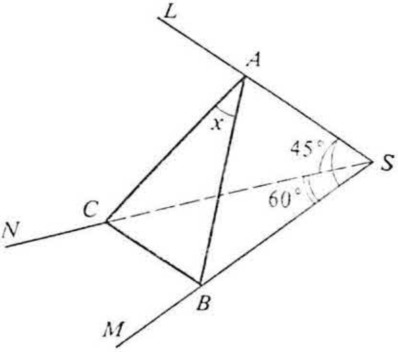


Fig. 131

Example 2. The base of a pyramid is a regular triangle. One of the lat-

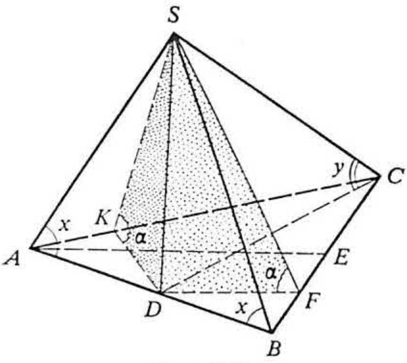


Fig. 132

eral faces is perpendicular to the plane of the base, two others being inclined to it at angles equal to α. Find the angles formed by the lateral edges and the plane of the base (Fig. 132).

Solution. Let the quadrilateral $SABC$ with its diagonals SB and AC be the representation of the given pyramid. This representation is complete (make this sure independently, taking, for instance, the plane ABC for a basic plane). Let us count the parametric number of the representation. Assuming an arbitrary triangle ABC to be the representation of the equilateral triangle, we spend two parameters. Assuming the triangle SAB to be the representation of the face, which is perpendicular to the plane of the base in the original, we spend one parameter. Finally, assuming the triangles SAC and SBC to be the representations of the faces inclined to the plane of the base at angles, each of which is equal to α, we spend one more parameter. Thus, the parametric number of the representation being considered $p = 4$.

To solve the problem, let us introduce the given and required quantities into the drawing, for which purpose we are going to carry out all necessary additional constructions.

First of all, we construct the line segment SD, i.e. the slant height of the face SAB. Then, since the plane SAB is perpendicular to the plane ABC, the line segment SD is also perpendicular to the plane ABC, and, therefore, AD is the projection of the edge SA on the plane ABC, that is, SAD is an angle formed by the edge SA and the plane ABC. Analogously, SBD is an angle made by the edge SB and the plane ABC, and SCD is an angle formed by the edge SC and the plane ABC. We further note that the median AE of the triangle ABC is also its altitude. Therefore, constructing DF parallel to AE, we conclude that DF is perpendicular to BC, and, consequently, DF is the projection of the line segment SF on the plane ABC. Thus, SF is perpendicular to BC, and $\angle SFD$ is the plane angle of the dihedral angle BC, that is, $\angle SFD = \alpha$. Analogously, on constructing $DK \perp AC$, we conclude that $\angle SKD = \alpha$. Moreover, the right triangles SDF and SDK have a common leg SD and the congruent acute angles SFD and SKD, that is, these triangles are congruent. Then $DF = DK$. Hence, the point D is equidistant from the sides of the angle ACB. Thus, the point D lies on the bisector of the angle ACB of the regular triangle ABC. But the bisector of an angle in a regular triangle serves as its median and altitude.

Thus, CD is a median of the triangle ABC, that is, the point D is the midpoint of the side AB, and, consequently, SD, i.e. the altitude of the triangle SAB, also serves as its median. This means that SAB is an isosceles triangle, that is, $\angle SBD = \angle SAD$.

Now, we may pass over to computations. Let us introduce an auxiliary parameter, putting $AB = a$, and set, for brevity, $\angle SBD = x$ and $\angle SCD = y$. Then $\angle SAD = x$ as well. It is clear that $BD = \frac{1}{2} AB = \frac{a}{2}$. Further, since AE is a median of the triangle ABC, and FD is a median of the triangle ABE, $DF = \frac{1}{2} AE = \frac{1}{2} \frac{a\sqrt{3}}{2} = \frac{a\sqrt{3}}{4}$. We have from the right triangle SDF: $SD = DF \tan\alpha = \frac{a\sqrt{3}}{4} \tan\alpha$. Thus, $\tan x = \frac{SD}{BD} = \frac{a\sqrt{3}}{2} \tan\alpha$, whence $x = \arctan\left(\frac{\sqrt{3}}{2}\tan\alpha\right)$. Further, $\tan y = \frac{SD}{CD} = \frac{1}{2}\tan\alpha$, whence $y = \arctan\left(\frac{1}{2}\tan\alpha\right)$.

Example 3. The dihedral angle at a lateral edge in a regular quadrangular pyramid is equal to α. Find the dihedral angle at an edge of the base of this pyramid (Fig. 133).

Solution. Let the figure $SABCD$ be the representation of the given pyramid. This representation is complete (make this sure independently). Let us count its parametric number. Assuming the parallelogram $ABCD$ to be the representation of the square, we spend two parameters. Let O be the point of intersection of the diagonals of the base. Assuming the line segment SO to be the representation of the altitude of the pyramid, we spend another two parameters. Finally, assuming the angle between lateral faces, e.g. SBC and SDC, to be the representation of the dihedral angle whose size is equal to α in the original, we spend one more parameter. Thus, all the five parameters have been spent on the representation of the given pyramid.

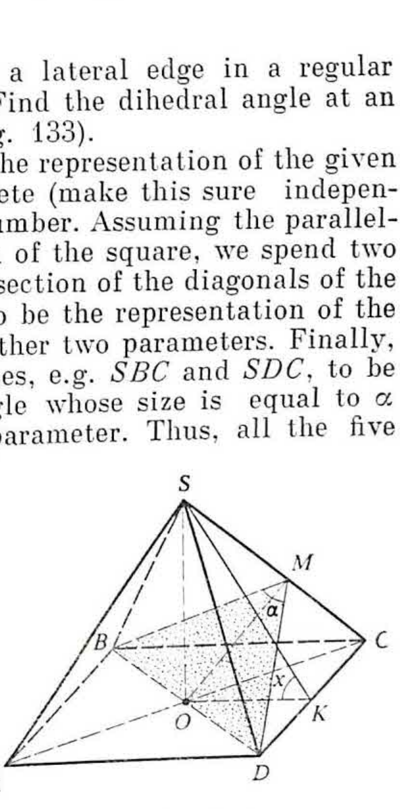


Fig. 133

To solve the problem, it is necessary to perform some additional constructions, enabling us, among other things, to introduce the given and required angles into the drawing. (It is impermissible, in principle, to perform arbitrarily new metric constructions on a given metrically determined representation. However, in the case under consideration, when making metric constructions, some exceptions are allowed, since the angle at a lateral edge of the pyramid is specified by a parameter whose values, as it is obvious, belong to the interval (90°, 180°). We are going to take advantage of this fact.)

Let us take a point M on the edge SC, and assume that the line segment OM is the representation of the perpendicular to the edge SC. We then construct the line segments DM and BM. Since $BD \perp AC$ and $BD \perp SO$, we have: $BD \perp SC$. Thus, $SC \perp BD$ and $SC \perp OM$, i.e. $SC \perp DM$ and $SC \perp BM$. Then $\angle BMD$ is the plane angle of the dihedral angle SC, and, therefore, $\angle BMD = \alpha$. Besides, we can show that the line segment OM, i.e. a median of the triangle BMD, also serves as its angle bisector and altitude. Therefore, $\angle BMO = \angle DMO = \frac{\alpha}{2}$ and $OM \perp BD$. Let us also construct SK, i.e. the slant height of the face SDC, and OK, i.e. a median of the triangle OCD. Since the triangles SDC and OCD are isosceles, $SK \perp CD$ and $OK \perp CD$, i.e. $\angle SKO$ is the plane angle of the dihedral angle at the edge CD.

Now, let us pass over to computations. Let us introduce an auxiliary parameter, putting $CD = a$, and set, for brevity, $\angle SKO = x$. Since $DO = \frac{a\sqrt{2}}{2}$ in the right triangle DOM, we have: $DM = \frac{a\sqrt{2}}{2\sin\frac{\alpha}{2}}$. Let us now compute SK. Since $\triangle SCK \sim \triangle CDM$, we have: $\frac{SK}{SC} = \frac{DM}{CD}$, where $SC = \sqrt{SK^2 + \frac{a^2}{4}}$ and $\frac{DM}{CD} = \frac{\sqrt{2}}{2\sin\frac{\alpha}{2}}$. Thus, $SK = \sqrt{SK^2 + \frac{a^2}{4}}\,\frac{\sqrt{2}}{2\sin\frac{\alpha}{2}}$, whence $SK = \frac{a}{2\sqrt{-\cos\alpha}}$. Then, we find from the right triangle SOK that $\cos x = \frac{OK}{SK} = \sqrt{-\cos\alpha}$, that is, $x = \arccos\sqrt{-\cos\alpha}$.

PROBLEMS TO BE SOLVED WITHOUT ASSISTANCE

690. Each of the two plane angles of a trihedral angle is equal to 45°, and the dihedral angle between them is equal to 90°. Find the third plane angle.

691. One of the plane angles of a trihedral angle is equal to 90°, and two others to 60° each. Find the angle between the plane of the right angle and the plane cutting off equal line segments from the edges of the trihedral angle.

692. Three plane angles of a trihedral angle are equal to 45°, 45°, and 60°. Drawn through its vertex is a straight line perpendicular to the face of the plane angle equal to 45°. Find the angle between this line and the edge of the trihedral angle not lying in the indicated face.

693. The faces SAB and SBC of a trihedral angle $SABC$ form a right angle, and two other dihedral angles are equal to α each. Find the plane angle ASC.

694. Drawn through a point taken on the edge of a dihedral angle equal to α in one of the faces is a ray forming an angle β with the edge, and in the other face, a ray perpendicular to the edge. Find the angle between these rays.

695. Inside a trihedral angle, all the plane angles of which are equal to 2α, drawn through the vertex of the angle is a ray inclined equally to the edges of a dihedral angle. Find the angle at which this ray is inclined to each edge.

696. The plane angles of a trihedral angle are equal to α, β, and γ, respectively. Find its dihedral angles.

697. In a rectangle $ABCD$, $AB:BC = a:b$. Drawn through the side AD is a plane P with which the diagonal of the rectangle makes an angle equal to 30°. Find the dihedral angle between the plane of the rectangle and the plane P.

698. Drawn in the plane P through the point A is an inclined line AB making an angle α with the plane P. Drawn through AB is the plane Q forming an angle β with the plane P. Find the angle between the straight line AB and the line of intersection of the planes P and Q.

699. Passed through the bisector of the right angle of a right triangle is a plane making an angle α with the plane of the triangle. Find the angles made by the legs of the triangle and this plane.

700. In a regular tetrahedron $SABC$, a plane is passed through the median AD of the base and K, i.e. the midpoint of the edge SB. Find the dihedral angle between this plane and the plane of the base.

701. In a regular triangular prism $ABCA_1B_1C_1$, all the edges are equal in length. A plane is passed through the edge AA_1 and D_1, i.e. the midpoint of the edge B_1C_1. Find the dihedral angle between this plane and the plane AD_1C.

702. In a regular triangular pyramid, the side of the base is one-third of the lateral edge. Find the dihedral angle at the lateral edge.

703. In a regular quadrangular pyramid, the lateral edge is inclined at an angle α to the plane of the base. Find the dihedral angle at the lateral edge.

704. The base of a pyramid is a square. Find the dihedral angles made by the lateral faces and the plane of the base, their ratios being 1:2:4:2.

705. In a regular triangular pyramid, the dihedral angle at the edge of the base is equal to α. Find the angle of inclination of the lateral edge to the plane of the base.

706. The angle at the lateral edge of a regular triangular pyramid is equal to α. Find the dihedral angle between the plane of the base and the lateral face of the pyramid.

707. In a regular triangular pyramid, the dihedral angle between each lateral face and the plane of the base is equal to α. Find the dihedral angle between the lateral faces.

708. Prove that if all the dihedral angles of a triangular pyramid are equal to one another, then all of its edges are also equal in length.

709. In a triangular pyramid $SABC$, the faces SAC and SAB are the right isosceles triangles with a common hypotenuse SA. The dihedral angle at the edge SA is equal to α. Find the dihedral angles at the edges SB and SC.

710. The base of the pyramid $SABCD$, all the lateral edges of which are equally inclined to the plane of the base, is a rectangle $ABCD$. A plane is passed through F, K, and L, i.e. the midpoints of the edges AB, AD, and SC. Find the dihedral angle formed by this plane and the plane of the base if $AB:AD:SA = 1:\sqrt{3}:2$.

711. The base of a pyramid is a square. Two lateral faces are perpendicular to the plane of the base, and two others make with it angles equal to α. Find the dihedral angle between the lateral faces not perpendicular to the plane of the base.

712. The altitude of a regular n-gonal pyramid is half the side of the base. Find the angle between the lateral face and the plane of the base.

713. In a regular n-gonal pyramid, the angle between a lateral edge and an adjacent edge of the base is equal to α. Find the dihedral angle formed by the lateral face and the plane of the base.

714. In a regular n-gonal pyramid, the dihedral angle at the lateral edge is equal to 2α. Find the angle at which the lateral edge is inclined to the plane of the base of the pyramid.

715. In a regular n-gonal pyramid, the dihedral angle at each lateral edge is equal to 2α. Find the dihedral angle at the edge of the base of the pyramid.

716. Two congruent rectangles $ABCD$ and ABC_1D_1 have a common side AB, and their planes form a dihedral angle equal to 60°. Find the angle between the diagonals AC and BD_1 if $AB:AD = 1:2$.

717. Two congruent rhombi $ABCD$ and ABC_1D_1 have a common side AB, and their planes form a dihedral angle equal to 45°. Find the angle between the sides BC and BC_1 of the rhombi if the acute angle of each of them is equal to 60°.

718. The parallelogram $ABCD$ in which AC is a smaller diagonal and $AB:AC:BC = 1:1:\sqrt{2}$ is bent along the diagonal AC so that the angle BAD becomes equal to 60°. Find the dihedral angle made by the planes of the triangles ABC and ADC.

719. In a parallelogram $ABCD$, the acute angle is equal to 60° and $AB:BC = 2:1$. Taken on the side AB is a point P and on the side CD a point Q, i.e. the midpoints of these sides. On bending the parallelogram along the straight line PQ, the planes PQC and PQD form a dihedral angle equal to 45°. Find the acute angle between the straight lines AP and BP.

720. In a regular triangular pyramid $SABC$, the dihedral angle at the edge of the base is equal to 45°. Find the angle between the median CD of the face SAC and the plane SAB.

721. The base of a quadrangular pyramid is a rhombus with an acute angle of 60°. Each lateral face is inclined at an angle of 45° to the plane of the base. Find the angles made by the slant height of the lateral face and the diagonals of the base.

722. The base of a pyramid $SABCD$ is a square $ABCD$, the lateral face SAB is perpendicular to the plane of the base and is an isosceles triangle, and each of the lateral edges SC and SD forms an angle of 45° with the plane of the base. Find the dihedral angle at the lateral edge SD.

723. Taken in the plane of the base of an equilateral cone (outside the base) is a point situated from the circle of the base at a distance equal to the radius of the base. Through this point, two tangent planes are drawn to the cone. Find the dihedral angle between them.

SEC. 12. SECTIONS OF POLYHEDRONS

We shall distinguish between two kinds of problems involving the necessity to represent sections:

(1) problems in which it is said that it is *required to construct a section*;

(2) problems in which it is said (or meant) that a *section has been already constructed*.

When solving problems belonging to the first kind, the construction of a section is accompanied by a description of the process of construction, which is carried out either according to the complete scheme for solving a construction problem (analysis, construction, proof, and investigation) or (in simple cases) according to a somewhat simplified scheme (for instance, analysis is omitted, construction is combined with proof). The investigation of a problem on constructing a section should not be confused with the investigation of the solution of a problem on computing certain quantities connected with a given section. Thus, the problem on constructing a diagonal section of a cube or a rectangular parallelepiped has six solutions, whereas the problem on computing the area of the diagonal section has only one solution in the case of a cube and three solutions in the case of a rectangular parallelepiped.

When solving problems belonging to the second kind, we proceed in a somewhat different way: if no more than five parameters are spent on the representation of a given polyhedron together with a section, then the construction of the section is not described (however, the section should be constructed for solving the problem), and if all the five parameters have been spent prior to constructing a section, then the construction required for representing the section is described in detail. This description is made in the same way as a standard description of any other constructions carried out on a metrically determined representation.

When solving problems of both kinds, it is necessary to make sure that the representation on which a section is to be constructed (for problems of the first kind) or on which a section is given (for problems of the second kind) is complete. (Recall that any position problem is permissible on a complete representation.) Besides, since, as a rule, we do not know in advance whether in the course of solution we shall succeed in confining ourselves to only position constructions or metric constructions as well, we have to compute the parametric number of the representation.

Let us pass over to considering problems of the first kind. We shall first dwell on constructing a section using the *method of the trace of a cutting plane*. (The *trace* of a cutting plane is defined as a straight line obtained by intersecting the cutting plane and a certain plane chosen on the representation as a basic one.)

Example 1. Given on the edges of the cube $ABCDA_1B_1C_1D_1$ are points P, Q, and R such that $AP=\frac{1}{3}AA_1$, $B_1Q=\frac{1}{2}B_1C_1$, and $CR=\frac{1}{3}CD$. Construct the section of the cube by the plane PQR (Fig. 134).

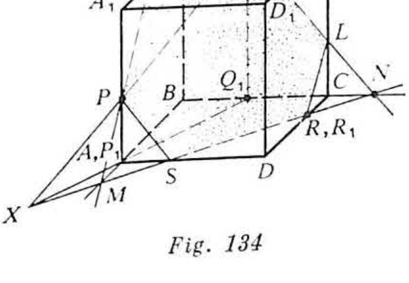


Fig. 134

Solution. Let us clarify whether this problem is solvable. Let the figure $ABCDA_1B_1C_1D_1$ be the representation of the cube. This is a complete representation. It is also clear that if the points P, Q, and R, i.e. the respective projections of the points P_0, Q_0, and R_0, are given on the representation, we can also find the secondary projections of the points P_0, Q_0, and R_0. For this purpose, it is sufficient to carry out in the plane of the representation the internal parallel projection, e.g. in the direction parallel to AA_1. Thus, we shall find the points P_1, Q_1, and R_1 and arrive at the conclusion that the representation of the cutting plane is a given one. Then the problem on finding the intersection of the plane given by the points P, Q, and R with the surface of the cube is solvable. It is not difficult to count that the parametric number $p = 5$, that is, all further constructions should be performed in accordance with the rules for parallel projection.

Let us pass over directly to constructing the section (although the construction of the representation of the section is meant). In the example under consideration, we shall omit the first step of the general scheme for solving a problem on construction (analysis), while the second and third steps (construction and proof) will be combined together.

Let us first of all find the trace of the cutting plane, i.e. the line of intersection of the planes PQR and ABC (the points P_1, Q_1, and R_1 are taken in the plane ABC, which is thus chosen as a basic one).

(1) Find the point X at which the lines PQ and P_1Q_1 intersect. Since the point X lies on the straight line PQ, and this line lies in the cutting plane PQR, the point X lies in the plane PQR. Analogously, since the point X lies on the straight line P_1Q_1, it lies in the plane ABC. Thus, the point X is a common point of the planes PQR and ABC. The point R is also a common point of these planes. Then XR is the line of intersection of the planes PQR and ABC.

(2) Construct the straight line XR, i.e. the trace of the cutting plane.

(3) Find the point S at which the straight lines RX and AD intersect.

(4) Join the points P and S.

(5) Find the point M at which the straight lines RX and AB intersect.

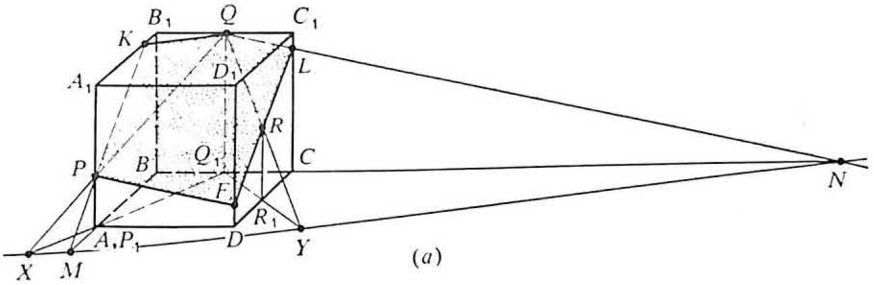
Since the point M lies on the straight line RX, and the latter lies in the cutting plane PQR, the point M lies in the plane PQR. Analogously, since the point M lies on the straight line AB, and this line lies in the plane ABB_1, the point M is a common point of the cutting plane and the plane of the lateral face ABB_1 of the cube.

(6) Construct the straight line MP of intersection of the planes PQR and ABB_1.

(7) Find the point K at which the straight line MP intersects the straight line A_1B_1.

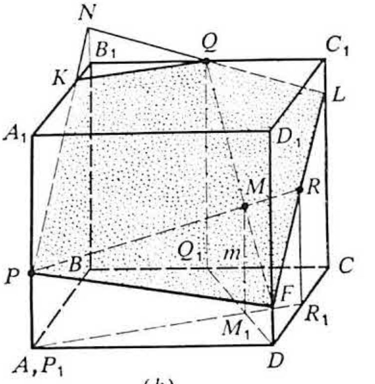
(8) Join the points K and Q.

(9) Find the point N of intersection of the straight lines RX and BC.



(a)

Fig. 135



(b)

(10) Construct the straight line QN.

(11) Find the point L at which the straight lines QN and CC_1 intersect.

(12) Join the point L to the point R.

Since, by construction, the vertices of the polygon $SPKQLR$ are points lying in the cutting plane PQR and belonging to the edges of the cube, the polygon $SPKQLR$ is the desired section. Since, according to the sense of the problem, the points P, Q, and R are noncollinear (do not lie in one straight line), the problem has a unique solution.

Example 2. Given a cube $ABCDA_1B_1C_1D_1$ and points P, Q, and R such that $AP = \frac{1}{3}AA_1$, $B_1Q = \frac{1}{2}B_1C_1$, and R is the centroid of the face DD_1C_1C. Construct the section of the cube by the plane PQR.

Solution. As in the preceding example, we find the points P_1, Q_1, and R_1 and show that the representation is complete, and $p = 5$. Finding the trace of the cutting plane (Fig. 135*a*), we get the desired section.

Using this example, we shall consider another method of construction of sections called the *method of internal projection*. (The completeness of the representation has been ascertained.) Let us now carry out relevant constructions (Fig. 135*b*).

(1) Construct the straight lines PR and P_1R_1.

(2) Find the point M_1 at which the straight lines P_1R_1 and Q_1D intersect.

(3) Through the point M_1 draw a straight line m parallel to AA_1.

(4) Find the point M of intersection of the straight lines m and PR.

(5) Find the point F of intersection of the straight lines QM and DD_1.

The point F lies in the cutting plane PQR. Indeed, the point M lies on the straight line PR, that is, the point M lies in the plane PQR. But the point Q also lies in the plane PQR. Hence, the straight line MQ lies in the plane PQR, and then the point F of the straight line MQ also lies in the plane PQR. After finding the fourth point belonging both to the cutting plane and to the plane of the section of the cube, all necessary constructions can be carried out in the following order:

(6) Join the points P and F.

(7) Find the point L of intersection of the lines FR and CC_1.

(8) Construct the straight line QL.

(9) Find the point N of intersection of the straight lines QL and BB_1.

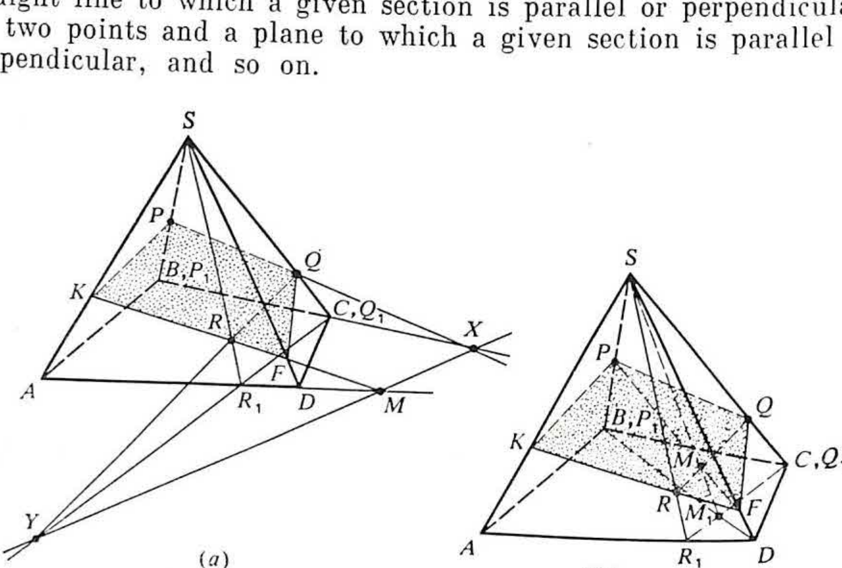
(10) Find the point K of intersection of the straight lines NP and A_1B_1.

(11) Join the points K and Q.

The polygon $PKQLF$ thus obtained is the required section.

The above-described methods of the trace of a cutting plane and of internal projection are also used for constructing the sections of a pyramid. In this case, central projection is realized. In Fig. 136*a* the section of the pyramid by the plane PQR is constructed by means of the trace XY of the cutting plane, while in Fig. 136*b* using the method of internal projection.

There is a variety of ways of representing the sections of polyhedrons. Thus, a cutting plane can be given by two points and a straight line to which a given section is parallel or perpendicular, by two points and a plane to which a given section is parallel or perpendicular, and so on.



(*a*) (*b*)

Fig. 136

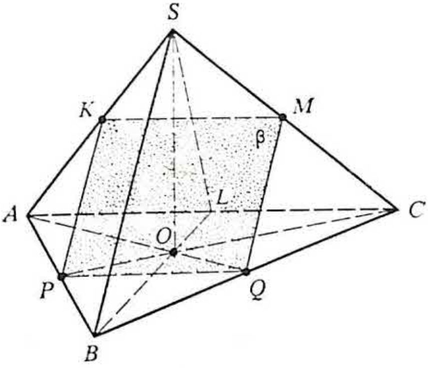


Fig. 137

Example 3. In a regular triangular pyramid $SABC$ draw a section parallel to the edge SB and passing through the points P and Q, i.e. the midpoints of the edges AB and BC, respectively (Fig. 137).

Solution. Let the quadrilateral $SABC$ with its diagonals AC and SB be the representation of the given pyramid. This representation is complete. The cutting plane is completely determined by two points P and Q and a straight line SB. Thus, the construction problem on this representation is solvable, and its parametric number $p = 4$.

Let us pass over to constructing the section. For the sake of brevity, we denote the cutting plane by β. Since the point P lies on the edge AB, it also lies in the plane ABC. Similarly, the point Q lies in the plane ABC. Again, the points P and Q also lie in the cutting plane β. Thus the planes β and ABC intersect along the straight line PQ. Thus, let us carry out relevant constructions.

(1) Join the points P and Q. Since SB is parallel to β, the plane SAB passing through the edge SB will intersect β along a straight line passing through the point P and parallel to the edge SB.

(2) Therefore, in the plane SAB, through the point P draw a straight line PK parallel to SB.

(3) Analogously, construct QM parallel to SB.

(4) Join the points K and M.

The quadrilateral $PKMQ$ meets the conditions of the problem, and, therefore, is the desired section. It is also not difficult to make sure that the required section exists and is unique.

Remark. The variety of methods of representing a cutting plane does not allow us to apply a certain unique universal method when constructing a section. Thus, in Examples 1 and 2 of this section it was possible to apply the methods of the trace of a cutting plane and of internal projection. In Example 3 neither of these methods would turn out to be effective.

Let, under conditions of Example 3, the side of the base of the pyramid be equal to a, and the lateral edge be equal to b. Find the area of the section. To this end, we have to determine the shape of the section (the kind of the quadrilateral $PKMQ$). Note that while constructing the section of the pyramid, we do not spend any parameters, that is, the parametric number of the representation of the pyramid with the section constructed is equal to 4. Assuming that $AB:SA = a:b$, we spend one more parameter on the represen-

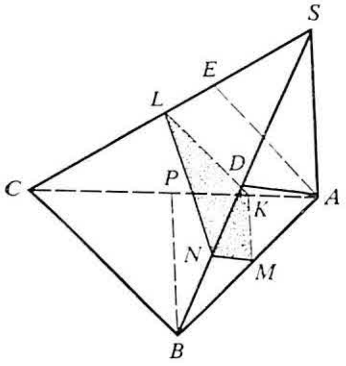


Fig. 138

tation. Thus, now $p = 5$. Consequently, no subsequent metric constructions are allowed to be carried out arbitrarily. However, all further constructions turn out to be positional.

Thus, since, by construction, $PK \parallel SB$ and $QM \parallel SB$, we have: $PK \parallel QM$. Besides, PK is the midline of the triangle SAB, that is, $PK = \frac{1}{2} SB = \frac{1}{2} b$. Analogously, $QM = \frac{1}{2} b$. Therefore, $PK = QM$, then the quadrilateral $PKMQ$ is a parallelogram, and $PK = \frac{b}{2}$, $PQ = \frac{a}{2}$. However, to compute the area of the parallelogram, these data are insufficient, and, therefore, we are going to define more exactly the shape of the parallelogram $PKMQ$. Let us construct BL, i.e. a median of the triangle ABC (no parameter is spent on this construction). The point O, i.e. the foot of the altitude SO of the pyramid, lies on the median BL. Since OL is perpendicular to AC, and OL is the projection of the line segment SL on the plane ABC, SL is perpendicular to AC. Thus, $AC \perp OL$ and $AC \perp SL$, but then $AC \perp SB$. (This proves that the skew edges of a regular triangular pyramid are mutually perpendicular.) Further, since $PQ \parallel AC$ and $PK \parallel SB$, we have: $PQ \perp PK$, that is, the parallelogram $PKMQ$ is a rectangle. Thus, we get: $S_{PKMQ} = \frac{ab}{4}$.

Example 4. The base of the pyramid $SABC$ is a right triangle ABC. The edge SA is perpendicular to the plane of the base, $SA = AB = BC = a$. Through the midpoint of the edge AB draw a cutting plane perpendicular to the edge SC and find the area of the section thus obtained (Fig. 138).

Solution. Let the quadrilateral $SABC$ with its diagonals SB and AC be the representation of the given pyramid. This representation is complete, and its parametric number $p = 5$ (make this sure independently). Let us pass straight to constructing the section.

(1) Construct the median AD of the triangle SAB.
(2) Find the point M, i.e. the midpoint of the edge AB.
(3) Draw in the plane SAB a straight line MN parallel to AD.
(4) Construct the median BP of the triangle ABC.
(5) Draw in the plane ABC a straight line MK parallel to BP.

To construct $KL \perp SC$, we first construct $AE \perp SC$. This is a metric construction, and, therefore, it is forbidden to be carried out arbitrarily. Let us proceed as follows. Note that in the right triangle ABC, $AB = BC = a$, and, therefore, $AC = a\sqrt{2}$. Then we find from the triangle SAC, where $SA = a$, that $SC = a\sqrt{3}$. Thus, for the line segment AE to be the representation of the perpendicular to the edge SC, the following equality must be fulfilled: $SA^2 - SE^2 = AC^2 - CE^2$, or $a^2 - SE^2 = 2a^2 - (a\sqrt{3} - SE)^2$, whence we find that $SE = \frac{a\sqrt{3}}{3}$, that is, $SE:SC = 1:3$.

Thus, the further constructions will be carried out in the following sequence:

(6) Find a point E such that $SE:SC = 1:3$.

(7) Join the points A and E.

(8) In the plane SAC construct KL parallel to AE.

(9) Join the points N and L.

Since the intersecting straight lines AE and AD define a plane, $LK \parallel AE$, and $MN \parallel AD$, the straight lines LK and MN also define a plane. Thus, $MKLN$ is a plane quadrilateral.

Let us prove that the plane MKL is perpendicular to the edge SC. Indeed, the straight line SA is perpendicular to the plane ABC, that is, $SA \perp MK$, or $MK \perp SA$. Besides, by construction, $MK \perp AC$. Then $MK \perp LK$ and $MK \perp SC$. Further, $SC \perp LK$ and $SC \perp MK$, that is, the straight line SC is perpendicular to the plane MKL.

Thus, the section $MKLN$ meets the conditions of the problem, and, consequently, is the desired one. Since the cutting plane is perpendicular to the given straight line and passes through the given point belonging to the surface of the pyramid, it is clear that the required section exists and is unique.

Thus, the construction of the representation has been completed, and we may pass over to finding S_{MKLN}, i.e. the area of the section. In order to compute the desired area, let us first determine the kind of the quadrilateral $MKLN$. We have from the right triangles ABP and SAB, respectively: $MK = \frac{1}{2} BP$ and $MN = \frac{1}{2} AD$. But $BP = AD = \frac{a\sqrt{2}}{2}$. Thus, $MK = MN = \frac{a\sqrt{2}}{4}$. Since the edge SA is perpendicular to the plane ABC, AB is the projection of the edge SB on the plane ABC. But $AB \perp BC$, then $SB \perp BC$ as well. From $\triangle SLN \sim \triangle SBC$, $\frac{LN}{BC} = \frac{SN}{NC}$, whence we get: $LN = \frac{a\sqrt{6}}{4}$. From $\triangle CLK \sim \triangle CAE$, $\frac{LK}{AE} = \frac{CK}{AC} = \frac{3}{4}$, whence we obtain: $LK = \frac{3}{4} AE$. But $AE \cdot SC = SA \cdot AC$, that is, $AE = \frac{a\sqrt{6}}{3}$, then $LK = \frac{a\sqrt{6}}{4}$.

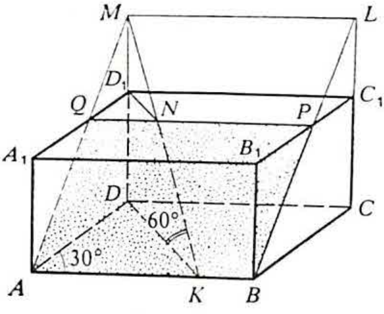


Fig. 139

Thus, the quadrilateral $MKLN$ possesses the following property: in it $MK = MN$ and $LK = LN$. Further, it is not difficult to reveal that $\triangle MKL = \triangle MNL$, and, therefore, $S_{MKLN} = 2S_{\triangle MKL}$. But, as was shown above, $MK \perp LK$. Consequently, $S_{\triangle MKL} = \frac{1}{2} MK \cdot LK = \frac{a^2\sqrt{3}}{16}$, and, hence, $S_{MKLN} = \frac{a^2\sqrt{3}}{8}$.

Consider several problems belonging to the second kind.

Example 5. The base of a right prism whose altitude is equal to 1 cm is a rhombus with sides equal to 2 cm each and an acute angle equal to 30°. Drawn through a side of the base is a cutting plane, the angle between which and the plane of the base is equal to 60°. Find the area of the section.

Solution. Let the figure $ABCDA_1B_1C_1D_1$ be the representation of the given prism (Fig. 139). This is a complete representation, and its parametric number $p = 5$ (make this sure independently).

Since the construction of the section in this example is a metric problem, and $p = 5$, it is impermissible, on having taken arbitrarily a point M on the straight line DD_1, and a point L on the straight line CC_1 (of course, such that $ML \parallel AB$), to state that the quadrilateral $AMLB$ is the representation of the given section. However, if we knew the position of the point M, i.e. the point of intersection of the cutting plane and the edge DD_1, we would construct easily the representation of the section.

Dropping the perpendicular DK from the point D to the edge AB, we find in the obtained right triangle ADK that $AD = 2$ cm, $DK = 1$ cm, and, consequently, $AK = \sqrt{3}$ cm. Thus, to construct DK perpendicular to AB, the point K should be chosen so that the equality $AK{:}AB = \sqrt{3}{:}2$ is fulfilled. This construction can be carried out, for instance, as follows: construct an auxiliary right triangle with hypotenuse AB and leg $\frac{1}{2}AB$. Then the other leg will be equal to $\frac{\sqrt{3}}{2}AB$. And since $\frac{\sqrt{3}}{2}AB{:}AB = \sqrt{3}{:}2$, AK may be assumed to be equal to the other leg of the auxiliary triangle. Further, since the edge DD_1 is perpendicular to the plane ABC, for any position of the point M on the edge DD_1, the line segment DK will serve as the projection of the line segment MK on the plane ABC, that is, the angle MKD will be the plane angle of the dihedral angle at the edge AB. If the point M is such that $\angle MKD = 60°$, then we find in the right triangle MKD that $DK = 1$ cm, $MK = 2$ cm, and $MD = \sqrt{3}$ cm. Thus, when constructing the point M, the equality $MD:DD_1 = \sqrt{3}:1$ should be necessarily taken into account. For instance, the point M can be constructed in the following way: construct an auxiliary right triangle with hypotenuse $2DD_1$ and leg DD_1. Then the other leg will be equal to $\sqrt{3}DD_1$. And since $\sqrt{3}DD_1:DD_1 = \sqrt{3}:1$, MD may be assumed to be equal to the other leg of the auxiliary triangle. On constructing the point M, draw ML parallel to AB, where the point L lies on the edge CC_1. Constructing the straight line AM, we find the point Q, i.e. the point of intersection of the straight lines AM and A_1D_1, and the point P in a similar way. The quadrilateral $AQPB$ is the representation of the given section. Now, we are going to compute S_{AQPB}.

It is obvious that the quadrilateral $AQPB$ is a parallelogram. Since MK is perpendicular to AB, NK is the altitude of this parallelogram, where N is the point of intersection of the straight lines MK and PQ. Thus, $S_{AQPB} = AB \cdot NK$. We find NK. We have in the right triangle MDK: $MK = 2$ cm and $MD = \sqrt{3}$ cm. It is easy to show that $DK \parallel D_1N$, i.e. $\frac{MK}{NK} = \frac{MD}{DD_1}$, or $\frac{2}{NK} = \frac{\sqrt{3}}{1}$, where $NK = \frac{2\sqrt{3}}{3}$ cm. Thus, $S_{AQPB} = \frac{4\sqrt{3}}{3}$ cm^2.

Example 6. In a regular triangular pyramid, a section is drawn through the midline of the lateral face parallel to a lateral edge and through the vertex of the base of the pyramid, the vertex not lying in this face. Determine the angle of inclination of the plane of the section to the plane of the base if it is known that the angle between each lateral edge of the pyramid and the plane of its base is equal to α.

Solution. Let the quadrilateral $SABC$ with its diagonals be the representation of the given pyramid (Fig. 140). This representatino is complete, and its parametric number $p = 5$. We construct points K and E, i.e. the midpoints of the edges BC and SB, and then triangle AEK, which is the representation of the given section of the pyramid (no parameters are spent on these constructions).

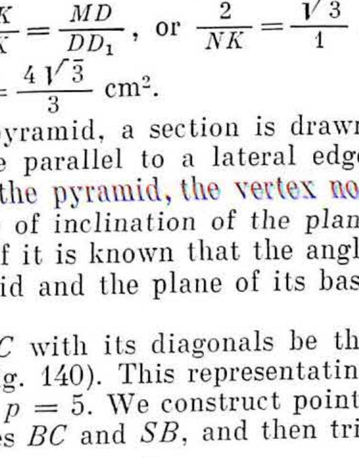


Fig. 140

Since SO is the altitude of the pyramid, $\angle SCO$ is equal to the angle made by the lateral edge and the plane of the base, that is, $\angle SCO = \alpha$. It is required to find the dihedral angle made by the cutting plane AEK and the plane of the base. Let us denote it by $\angle AK$. To determine the desired dihedral angle, we have to construct and find its plane angle. Since $p = 5$, no metric constructions are allowed to be carried out arbitrarily on the representation available. It is necessary to construct the representations of two perpendiculars to the edge of the dihedral angle. Let us carry out this construction in the following way. Join the point O to the point B and then construct EF parallel to SO in the plane SOB. Then, since SO is perpendicular to the plane ABC, EF is also perpendicular to this plane and then EF is perpendicular to OB. Since AK is a median of the equilateral triangle ABC, we have: $BC \perp AK$. Draw FL parallel to BC, then FL is perpendicular to AK. Finally, construct EL. Thus, $AK \perp FL$ and $AK \perp EF$, but then AK is perpendicular to EL, that is, $\angle ELF$ is the plane angle of the desired dihedral angle. We set $\angle ELF = x$. To determine x, let us introduce an auxiliary parameter $AB = a$. Now, we are going to find EF and FL. Since EK is the midline of the triangle SBC, we have: $BE = SE$. But, by construction, EF is parallel to SO. Consequently, EF is the midline of the triangle SOB, and, therefore, $EF = \frac{1}{2} SO$. We find from the right triangle SOC that $SO = CO \tan\alpha$, where $CO = \frac{2}{3}\ \frac{a\sqrt{3}}{3} = \frac{a\sqrt{3}}{3}$. Thus, $EF = \frac{a\sqrt{3}}{6}\tan\alpha$. Further, we find that $FL = \frac{1}{2} BK = \frac{1}{4} a$. Thus, $\tan x = \frac{EF}{FL} = \frac{2\sqrt{3}}{3}\tan\alpha$, and, consequently, $x = \arctan\left(\frac{2\sqrt{3}}{3}\tan\alpha\right)$.

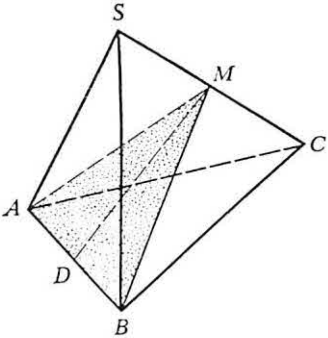


Fig. 141

Remark. In the example under consideration, it may seem that, at first glance, it is more expedient to construct the plane angle of the desired dihedral angle in the following way (do this independently): (1) $OM \parallel BC$; (2) SM; (3) P, i.e. the point of intersection of SM and AE; (4) OF. Actually, in such a construction, $\angle POM$ is the plane angle of the dihedral angle formed by the planes AEK and ABC. However, all the attempts to determine the angle POM thus constructed meet considerable difficulties.

Sometimes, we have to construct a section when solving such problems in which nothing is said about sections at all (see Example 2, Sec. 10). In some cases, the construction of a section is not necessary, in principle, whereas the solution of the problem can be simplified by means of it.

Example 7. One edge of a triangular pyramid is equal to a, and each of the remaining edges is equal to b. Find the volume of the pyramid.

Solution. Without dwelling on the construction of the representation (Fig. 141) and other steps of the solution, we should like to note that if we decided to solve this problem by means of the formula $V = \frac{1}{3} S_{\text{base}}H$, then, to find the altitude H, we would have to perform rather complicated computations. Therefore, let us take another way. Let us construct the section of the pyramid by the plane passing through the edge AB perpendicular to the edge SC. If $AB = a$, then $V = \frac{1}{3} S_{\Delta ABM}SM + \frac{1}{3} S_{\Delta ABM}CM = \frac{1}{3} S_{\Delta ABM}SC = \frac{b}{3} S_{\Delta ABM}$.

(Note that in this case the construction of a section is a position problem, since the triangles SAC and SBC are equilateral, that is, the medians AM and BM are the representations of the perpendiculars to the edge SC, and, therefore, the triangle ABM is the representation of the section perpendicular to the edge SC.)

Since $S_{\Delta ABM} = \frac{1}{2} AB \cdot MD$, where MD is a median of the isosceles triangle ABM in which $AM = BM = \frac{b\sqrt{3}}{2}$, we have: $S_{\Delta ABM} = \frac{1}{2} a \sqrt{\frac{3b^2}{4} - \frac{a^2}{4}} = \frac{a}{4}\sqrt{3b^2 - a^2}$. Consequently, $V = \frac{ab}{12}\sqrt{3b^2 - a^2}$.

PROBLEMS TO BE SOLVED WITHOUT ASSISTANCE

724. In a cube $ABCDA_1B_1C_1D_1$ whose edge is equal to a, F is the midpoint of the edge D_1C_1. Find the distance from the points A_1, A, and C_1 to the plane passing through the points B, F, and D.

725. A cutting plane is passed through the diagonal of the lower base of a cube whose edge is equal to a and the midpoint of one of the sides of the upper base. Find the distance from the centre of the cube to this plane.

726. In a regular quadrangular pyramid draw a plane through the diagonal of the base parallel to the lateral edge. Find the area of the section thus obtained if the side of the base and the lateral edge are equal to a and b, respectively.

727. Each edge of a regular quadrangular pyramid is equal to a. Construct the section of the pyramid by the plane passing through the midpoints of two adjacent sides of the base and the midpoint of the altitude. Find the area of this section.

728. In a regular hexagonal prism whose lateral faces are squares with side a construct the section of the prism by the plane passing through the side of the lower base and the opposite side of the upper base. Find the area of this section.

729. In a right parallelepiped, the acute angle of the base is equal to α. The section of the parallelepiped, which is passed through the side of the base whose length is equal to a and the opposite edge, has the area S and makes an angle equal to $90° - \alpha$ with the plane of the base. Find the length of the other side of the base.

730. The altitude of a regular triangular prism is equal to H. A plane is passed through one of the edges of the base and the opposite vertex of the other base. Find the area of the triangle obtained in the section if its angle at the indicated vertex is equal to 2α.

731. In a cube $ABCDA_1B_1C_1D_1$, a cutting plane is passed through the points P and Q, i.e. the midpoints of the respective edges AB and AD, and the vertex C_1. Find the distance from the vertex C to the cutting plane if the edge of the cube is equal to a.

732. A cutting plane is passed through the vertex A of the base of a cube $ABCDA_1B_1C_1D_1$ and the points P and Q, i.e. the midpoints of the respective edges B_1C_1 and C_1D_1. Find the area of the section if the edge of the cube is equal to a.

733. A cutting plane is passed through the point K taken on the edge AA_1 of a cube $ABCDA_1B_1C_1D_1$ and the points P and Q, i.e. the midpoints of the respective edges B_1C_1 and C_1D_1. Find the area of the section if the edge of the cube is equal to a and $AK:KA_1 = 1:2$.

734. A cutting plane is passed through the points P and Q, i.e. the midpoints of the respective edges AA_1 and CC_1 of a regular triangular prism $ABCA_1B_1C_1$, and a point D taken on the lateral edge BB_1 so that $B_1D:BD = 3:2$. Find the area of the section if each edge of the prism is equal to a.

735. The base of a pyramid, each lateral edge of which is equal to $a\sqrt{3}$, is a rectangle $ABCD$ with sides equal to a and $2a$. Construct the section of the pyramid by the plane passing through the diagonal BD of the base parallel to the lateral edge SA. Find the area of the section.

736. The lower base of a prism is a rhombus $ABCD$, the vertex angle of which is equal to 60°. The vertex A_1 of the upper base is equidistant from the vertices A, B, and D, the edge $AA_1 = l$ and makes an angle α with the plane of the base. Construct the section of the prism by the plane passing through the diagonal A_1C parallel to the diagonal BD. Find the area of the section.

737. The base of a pyramid is an isosceles triangle with a lateral side equal to a. The angle at the base of the triangle is equal to α, and each lateral edge is inclined at an angle β to the plane of the base. Construct the section of the pyramid by the plane passing through the altitude of the pyramid and the vertex of one of the angles equal to α. Find the area of the section.

738. Drawn in a regular tetrahedron are two sections each of which is parallel to the edges AB and SC. The area of the part of the face SAC enclosed between the cutting planes is by S cm^2 greater than the area of the part of the face SAB enclosed between these planes. By how much is the area of one section greater than the area of the other?

739. The area of the base of a rectangular parallelepiped is equal to S. A plane is passed through the vertex A_1 of the upper base $A_1B_1C_1D_1$. The plane intersects the lateral edges BB_1, CC_1, and DD_1 at points B_2, C_2, and D_2, respectively. Find the volume of the part of the parallelepiped situated under the cutting plane if it is known that $CC_2 = c$, and the altitude of the parallelepiped is equal to H.

740. The base of a pyramid $SABCD$ is a rhombus $ABCD$ in which $AC = a$ and $BD = b$. The lateral edge SA whose length is c is perpendicular to the plane of the base. Through the point A and the midpoint of the edge SC draw a plane parallel to the diagonal BD. Find the area of the section thus obtained.

741. One side of the base of a regular quadrangular pyramid is equal to a. A section is drawn through the side of the base and the middle of the skew lateral edge. Find the distance from the plane of the section to the vertex of the pyramid if its altitude is equal to H.

742. Drawn in a regular triangular prism through one of the sides of the base is a plane forming an angle α with the plane of the base. Find the area of the triangular section thus obtained if it is known that the side of the base is equal to a.

743. The angle between a lateral face of a regular quadrangular pyramid and the plane of its base is equal to α. The slant height of the lateral face is equal to a. Drawn through one of the sides of the base is the section of the pyramid making an angle β with the plane of the base. Find the area of the section.

744. One side of the base of a regular quadrangular pyramid is equal to a. The angle between the lateral edge and the altitude of the pyramid is equal to 30°. Construct the section of the pyramid by the plane passing through the vertex of the base perpendicular to the opposite edge. Find the area of the section.

745. The base of a prism is a square $ABCD$ whose vertices are equidistant from the vertex A_1 of the upper base, $AA_1 = a$, and the angle between the lateral edge AA_1 and the plane of the base is equal to 60°. Construct the section of the prism by the plane perpendicular to the edge AA_1 and passing through the vertex C. Find the area of the section.

746. In a regular triangular pyramid, the side of the base is equal to a. The angle between two adjacent lateral edges is equal to 2α. Construct the section of the pyramid by the plane passing through one of the sides of the base perpendicular to the opposite lateral edge. Find the area of the section.

747. In a regular triangular pyramid, drawn through the edge of the base whose length is equal to a is a section perpendicular to the opposite lateral edge. Find the surface area of the pyramid if the cutting plane divides the lateral edge in the ratio $m:n$.

748. In a triangular pyramid $SABC$, the edge SA is perpendicular to the plane ABC, $AC = BC = a$, and $AS = AB = a\sqrt{2}$. Drawn through the midpoint of the edge AC is a plane perpendicular to the edge SB. Find the distance from the vertex A to this plane.

749. In a cube $ABCDA_1B_1C_1D_1$ construct the section passing through the points B, M, i.e. the midpoint of the edge CC_1, and K, i.e. the midpoint of the edge AD. Find the dihedral angle between the plane of the section and the plane $ABCD$.

750. Construct the section of a rectangular parallelepiped $ABCDA_1B_1C_1D_1$ by the plane passing through the vertex A, the midpoint of the edge CD, and the centroid of the face BCC_1B_1. Find the dihedral angle between the plane of the section and the plane $ABCD$ if $AB:AD:AA_1 = 2:3:4$.

751. Construct the section of a regular triangular prism $ABCA_1B_1C_1$ by the plane passing through the vertex A, the point K, i.e. the midpoint of the edge BB_1, and the point M of the edge CC_1 if $CM:C_1M = 1:2$ and $AB:BB_1 = 1:3$. Find the dihedral angle between the plane of the section and the plane ABC.

752. Construct the section of a regular quadrangular pyramid $SABCD$ (S the vertex) by the plane passing through the point A, the point P, i.e. the midpoint of the altitude SO, and the point K of the edge SD if $SK:KD = 2:1$ and $SB = BD$. Find the dihedral angle between the plane of the section and the plane of the base.

753. The angle between each lateral edge and the plane of the base in a regular triangular pyramid $SABC$ (S the vertex) is equal to α. Construct the section of this pyramid by the plane passing through the point A, the point P, i.e. the midpoint of the altitude SO, and the point K of the slant height SD of the face SAC if $SK:KD = 2:1$. Find the dihedral angle between the plane of the section and the plane of the base.

754. A regular quadrangular prism is cut by a plane so that a rhombus is obtained in section. The acute angle of the rhombus is equal to 2α. Find the dihedral angle between the cutting plane and the plane of the base.

755. The base of a pyramid is an isosceles triangle, the angle between the equal sides of which is equal to α. Each lateral edge of the pyramid is inclined at an angle β to the plane of the base. A cutting plane is drawn in this pyramid passing through the altitude of the pyramid and the vertex of the angle equal to α. Find the ratio of the area of the obtained section to the area of the base of the pyramid.

756. The base of an oblique parallelepiped is a rhombus $ABCD$ in which $\angle BAD = 60°$. Each lateral edge of the parallelepiped forms an angle equal

to 60° with the plane of the base, and the plane AA_1C_1C is perpendicular to the plane of the base. Find the ratio of the area of the section BB_1D_1D to the area of the section AA_1C_1C.

757. The base of a right parallelepiped is a parallelogram, the ratio of whose sides is $AB:BC=1:2$, and the angle at the vertex B is equal to 120°. Through the point D and the opposite vertex of the upper base draw a cutting plane parallel to the diagonal AC. Find the angle made by this plane and the plane of the base if the ratio of the altitude of the parallelepiped to the smaller side of the base is equal to $\sqrt{3}:1$.

758. The base of a pyramid is a rectangle $ABCD$, and the altitude of the pyramid is projected in the vertex B of the base. The lengths of the sides of the rectangle and the altitude of the pyramid are as 2:3:5. Through the diagonal BD draw a plane parallel to one of the edges of the pyramid not intersecting the diagonal BD. Find the angle of inclination of the cutting plane to the plane of the base of the pyramid.

759. A regular triangular pyramid is cut by a plane passing through one of the vertices of the base and the midpoints of two lateral edges. Find the ratio of the area of the lateral surface of the pyramid to the area of the base if it is known that the cutting plane is perpendicular to one of the lateral faces.

760. In a regular quadrangular prism, two parallel sections are drawn: one passes through the midpoints of two adjacent sides of the base and the midpoint of the axis of symmetry of the prism, and the other divides the axis in the ratio 1:3. Find the ratio of the areas of the first and second sections.

761. The base of a pyramid is a regular triangle with side a. One of the faces of the pyramid is perpendicular to the plane of the base, and this face is an isosceles triangle with lateral side equal to b. Find the area of the section of the pyramid which is a square.

762. A cutting plane bisects the dihedral angle at the base of a regular quadrangular pyramid. Find the area of the section if the side of the base of the pyramid is equal to a, and the dihedral angle at the base to 2α.

SEC. 13. SURFACES

Example 1. The base of a right prism $ABCA_1B_1C_1$ is a triangle in which $AB = AC$ and $\angle ABC = \alpha$. It is also known that D is the midpoint of the edge AA_1, $\angle DCA = \beta$, and $CD = b$. Find the lateral surface area of the prism (Fig. 142).

Solution. Let the figure $ABCA_1B_1C_1$ be the representation of the given prism. This representation is complete. Let us find its parametric number. Assuming that AB and AC are the representations of the congruent line segments, we spend one parameter. Assuming that the angle ABC is the representation of the angle equal to the angle α in the original, we spend one more parameter. Assuming that the line segment AA_1 is the representation

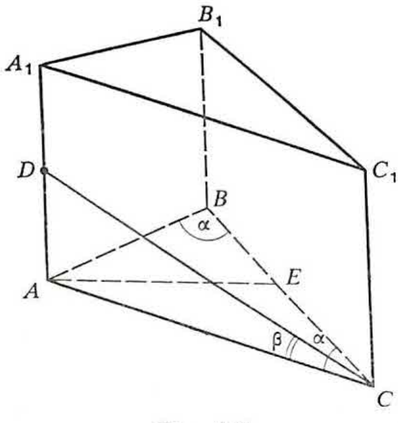


Fig. 142

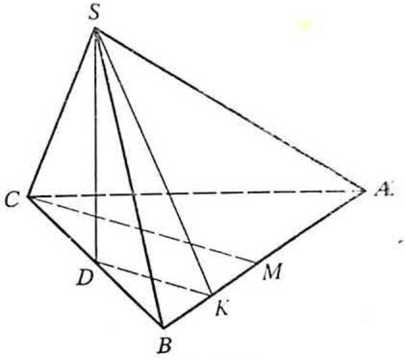


Fig. 143

of the perpendicular to the plane ABC, we spend two parameters. Finally, assuming that the angle DCA is the representation of the angle, which is equal to the angle β in the original, we spend one more parameter. Thus, the parametric number $p = 5$.

Let us construct on this representation the midpoint of the edge AA_1 (point D) and assume that $CD = b$ (no parameter is spent on this construction). Now, we are going to carry out all necessary computations. Since $ABCA_1B_1C_1$ is a right prism, $S_{\text{lat}} = P \cdot H$, where $P = 2AC + BC$ and $H = AA_1$. Since ACD is a right triangle, we have: $AC = b \cos\beta$, $AD = b \sin\beta$, and, therefore, $AA_1 = 2b \sin\beta$. We construct AE, i.e. a median of the triangle ABC, $AE \perp BC$. We find from the right triangle ACE that $CE = AC \cos\alpha = b\cos\beta\cos\alpha$, that is, $BC = 2b \cos\alpha \cos\beta$. Hence, $S_{\text{lat}} = (2b\cos\beta + 2b\cos\alpha\cos\beta)\, 2b\sin\beta = 4b^2\cos\beta\,(1+\cos\alpha)\sin\beta = 4b^2 \sin 2\beta \cos^2\frac{\alpha}{2}$. Thus, $S_{\text{lat}} = 4b^2 \sin 2\beta \cos^2\frac{\alpha}{2}$.

Example 2. The base of a pyramid is an equilateral triangle with side a. One of the lateral faces, which is perpendicular to the plane of the base, is also an equilateral triangle. Find the lateral surface area of the pyramid.

Solution. Let the quadrilateral $SABC$ with its diagonals be the representation of the given pyramid (Fig. 143). This is a complete representation, and its parametric number $p = 5$. Assuming that the triangle ABC is the representation of an equilateral triangle, we spend two parameters. Assuming that the triangle SBC is the representation of an equilateral triangle, we also spend two parameters. Finally, assuming that the dihedral angle at the edge BC is the representation of a right dihedral angle, we spend one more parameter.

Since the pyramid is not regular, we find its lateral surface area (S_{lat}) as the sum of the areas of the lateral faces: $S_{\text{lat}} = S_{\Delta SAB} + S_{\Delta SAC} + S_{\Delta SBC}$. But $\triangle SAB = \triangle SAC$ (by three sides), that is, $S_{\text{lat}} = 2S_{\Delta SAB} + S_{\Delta SBC}$, $S_{\Delta SAB} = \frac{1}{2} AB \cdot SK$, where SK is the altitude of the triangle SAB, i.e. $SK \perp AB$.

In the general case, the construction of a perpendicular to a given straight line is, as is known, a metric construction. In the example under consideration, it is required to carry out this construction on a metrically determined representation.

We construct the altitude SK in the following way:

(1) Construct SD, i.e. a median of the triangle SBC.

(2) Construct CM, i.e. a median of the triangle ABC.

(3) Through the point D draw DK parallel to CM.

(4) Join the point S to the point K.

Since SD is a median of the equilateral triangle SBC, we have: $SD \perp BC$. But the plane SBC is perpendicular to the plane ABC. Then SD is perpendicular to the plane ABC, and, consequently, DK is the projection of the line segment SK on the plane ABC. But DK is parallel to CM, and CM is a median of the equilateral triangle ABC, that is, $CM \perp AB$, and, therefore, $DK \perp AB$. Thus, $SK \perp AB$. We find from the right triangle SDK that $SK = \sqrt{SD^2 + DK^2}$, where $SD = \frac{a\sqrt{3}}{2}$ and $DK = \frac{1}{2} CM = \frac{a\sqrt{3}}{4}$. Thus, $SK = \frac{a\sqrt{15}}{4}$. We get: $S_{\triangle SAB} = \frac{a^2\sqrt{15}}{8}$, $S_{\triangle SBC} = \frac{a^2\sqrt{3}}{4}$, and, therefore, $S_{\text{lat}} = \frac{a^2}{4}(\sqrt{15} + \sqrt{3})$.

Example 3. The base of a pyramid is an isosceles trapezoid whose parallel sides are equal to a and b $(a > b)$. Each lateral face is inclined at an angle α to the base. Find the total surface area of the pyramid (Fig. 144).

Solution. Let the figure $SABCD$ be the representation of the given pyramid. This is a complete representation. Let us find its parametric number. Assuming the quadrilateral $ABCD$ to be the representation of the given isosceles trapezoid, we spend two parameters. Assuming SO to be the representation of the perpendicular to the plane ABC, we also spend two parameters. Finally, assuming the angles AB, BC, CD, and AD to be the representations of the dihedral angles, each of which is equal to α in the original, we spend only one parameter.

Indeed, suppose that the line segments OM, OL, and ON are the representations of the perpendiculars to the respective sides AD, DC, and BC of the trapezoid. Then it is clear that the line segments SM, SL, and SN will be the representations of the perpendiculars to the same sides AD, DC, and BC, respectively. Therefore, the angles SMO, SLO, and SNO will be the representations of the plane angles of the dihedral angles AD, DC, and BC, respectively. But then the triangles SMO, SLO, and SNO will be the representations of the congruent triangles, and the line seg-

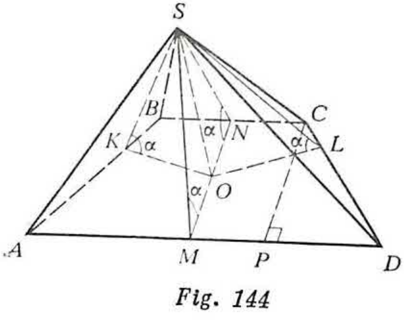


Fig. 144

ments OM, OL, and ON the representations of the line segments congruent in the original. Also, if OK is the representation of the perpendicular to the side AB, then OK is the representation of the line segment whose original has the same length as the original of the line segment OM.

Thus, O is the representation of the point equidistant from all the sides of the trapezoid. In other words, the point O is the representation of the centre of the circle inscribed in the base of the given pyramid. And this means that, firstly, M and N are the midpoints of the bases AD and BC of the trapezoid, and, secondly, the line segments DM and DL are the representations of the line segments congruent in the original, and the line segments CN and CL are also the representations of the congruent line segments. But then $DM:CN = DL:CL$. And since $DM:CN = \frac{a}{2}:\frac{b}{2}$, i.e. $DM:CN = a:b$, we also have $DL:CL = a:b$. Analogously, $AK:BK = a:b$.

Thus, no parameter is spent on the construction of the perpendiculars OM, OL, ON, and OK to the sides of the trapezoid. And this means that, assuming, for instance, $\angle SMO$ to be the representation of the angle equal to α in the original, we spend only one parameter, and assuming further that the angles SLO, SNO, and SKO are also equal to α, we spend no parameter.

Consequently, five parameters have been spent on the representation of the given pyramid.

Let us now compute the total surface area of the pyramid, S_{tot}. We have: $S_{\text{tot}} = S_{\text{lat}} + S_{ABCD}$. Since the right triangles SMO, SLO, SNO, and SKO are congruent, we get: $SM = SL = SN = SK$. Then $S_{\text{lat}} = \frac{1}{2}(AD + CD + BC + AB)\, SM = \frac{1}{2}(a + b + 2CD)\, SM$. But $DL = DM = \frac{a}{2}$ and $CL = CN = \frac{b}{2}$, i.e. $CD = \frac{a+b}{2}$. Construct $CP \parallel MN$. Then $CP \perp AD$. We find from the right triangle CDP, where $CD = \frac{a+b}{2}$ and $DP = \frac{a-b}{2}$, that $CP = \sqrt{\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2} = \sqrt{ab}$. Consequently, $OM = \frac{\sqrt{ab}}{2}$. We have from the right triangle SOM: $SM = \frac{OM}{\cos\alpha} = \frac{\sqrt{ab}}{2\cos\alpha}$. Thus, $S_{\text{lat}} = \frac{1}{2}(a + b + (a+b))\frac{\sqrt{ab}}{2\cos\alpha} = \frac{(a+b)\sqrt{ab}}{2\cos\alpha}$. Further, $S_{ABCD} = \frac{1}{2}(a+b)\sqrt{ab}$. Thus, $S_{\text{tot}} = \frac{(a+b)\sqrt{ab}}{2\cos\alpha} + \frac{(a+b)\sqrt{ab}}{2} = \frac{(a+b)\sqrt{ab}\cos^2\frac{\alpha}{2}}{\cos\alpha}$.

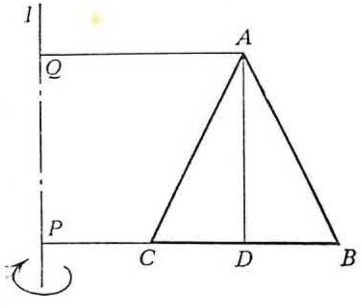


Fig. 145

Example 4. A regular triangle whose side is equal to a revolves about an external axis parallel to its altitude and $\frac{3}{2}a$ distant from it. Find the surface area of the solid of revolution thus obtained.

Solution. Since the axis of revolution l_0 is parallel to the altitude of the triangle, it is also parallel to the plane of the triangle, and, therefore, the section of the given solid of revolution by the plane passing through l_0 will be represented by the figure Φ_0 consisting of a pair of regular triangles symmetrical with respect to l_0.

Thus, in the example under consideration, we may confine ourselves to representing a plane figure instead of representing a rather complicated solid of revolution, that is, we may construct the representation of the indicated figure Φ_0 (similar to the original). Moreover, to solve the problem, we may even confine ourselves to representing the axis l_0 and only one of the triangles obtained in the section of the given solid of revolution by the plane passing through l_0 (Fig. 145). Thus, the triangle ABC is regular, $AB = a$, AD is the altitude of the triangle ABC, $l \parallel AD$, $DP \perp l$, and $DP = \frac{3a}{2}$. It is required to find the surface area of the solid of revolution.

Let us denote this desired area by S_{ABC}. We also denote the areas of the surfaces generated by revolving the line segments AB, AC, and BC about the axis l by S_{AB}, S_{AC}, and S_{BC}, respectively. Then $S_{ABC} = S_{AB} + S_{AC} + S_{BC}$. We have: $S_{AB} = \pi(BP + AQ)AB$. But $BP = BD + DP = 2a$, $AQ = DP = \frac{3}{2}a$, and $AB = a$. Thus, $S_{AB} = \frac{7}{2}\pi a^2$. Analogously, $S_{AC} = \pi(CP + AQ)AC = \frac{5}{2}\pi a^2$. And, further, $S_{BC} = \pi BP^2 - \pi CP^2 = \pi(4a^2 - a^2) = 3\pi a^2$. Thus, $S_{ABC} = 9\pi a^2$.

Remark. The desired surface area can be computed more easily if we take advantage of the first Guldin's theorem, according to which $S = P2\pi R$, where P is the perimeter of the figure revolving about the axis, and R the distance from the centroid of this figure to the axis of revolution. In the example under consideration, $P = 3a$ and $R = \frac{3}{2}a$.

Example 5. In a regular quadrangular pyramid, the angle between two adjacent lateral faces is equal to 2α. Find the ratio of the area of the diagonal section of the pyramid to the area of its lateral surface (Fig. 146).

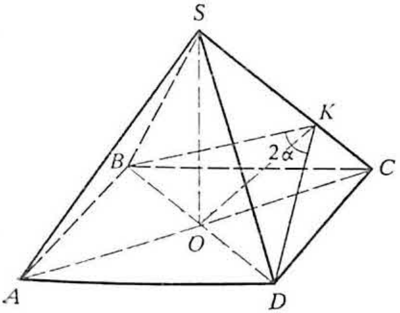


Fig. 146

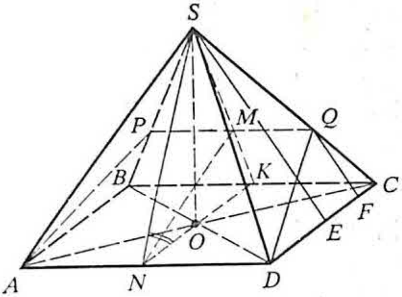


Fig. 147

Solution. Let the figure $SABCD$ be the representation of the given pyramid. This is a complete representation, and its parametric number $p = 5$ (make this sure independently). Thus, the dihedral angle at the edge SC, that is, $\angle SC = 2\alpha$. It is required to find the ratio $S_{\Delta SAC}:S_{\text{lat}}$.

To carry out all necessary computations, let us first construct the representation of the plane angle, e.g. at the lateral edge SC. For this purpose, it is required to drop a perpendicular from the point D to the edge SC. Let DK be the representation of the perpendicular to the edge SC. Thus, one more parameter is spent on the representation, and we are dealing here with overdetermined (and true) representation. We then construct BK and OK. Then the angle BKD is the plane angle of the dihedral angle SC, and, consequently, $\angle BKD = 2\alpha$. It is not difficult to prove that $BK = DK$, and, therefore, OK, i.e. a median of the triangle BDK, is its angle bisector and altitude. Thus, $\angle DKO = \alpha$. Note that $S_{\Delta SAC} = 2S_{\Delta SOC} = OK \cdot SC$ and $S_{\text{lat}} = 4S_{\Delta SCD} = 2DK \cdot SC$. Thus, $\frac{S_{\Delta SAC}}{S_{\text{lat}}} = \frac{1}{2}\frac{OK}{DK}$. But we have from the right triangle ODK: $\frac{OK}{DK} = \cos\alpha$, and then we get: $\frac{S_{\Delta SAC}}{S_{\text{lat}}} = \frac{\cos\alpha}{2}$.

Example 6. In a regular quadrangular pyramid, a cutting plane drawn through a side of the base bisects the lateral surface and the dihedral angle at the edge of the base. Find the angle between each lateral face and the plane of the base.

Solution. Let the figure $SABCD$ (Fig. 147) be the representation of the given pyramid. This is a complete representation, and its parametric number $p = 4$ (make this sure independently). Given in the pyramid is a section meeting the metric conditions. Let us count how the parameters are spent on the representation of this section.

We note first of all that since the section passes through an edge of the base (say, through the edge AD), and since AD is parallel to BC, AD will also be parallel to the plane SBC. Then the cutting plane intersects the face SBC along PQ ($PQ \parallel AD$). Further, we carry out some other constructions. We construct the point M, i.e. the midpoint of the line segment PQ, the straight line SM, and the point K, i.e. the point of intersection of the straight lines SM and BC. It is clear that $BK = CK$. We also construct the straight line KO and the point N, i.e. the point of intersection of the straight lines OK and AD. Since OK is parallel to AB, N is the midpoint of the edge AD. We now construct SN and NM. Since N is the midpoint of the edge AD, we have: $ON \perp AD$. But ON is the projection of the line segment SN on the plane ABC. Consequently, SN is perpendicular to AD. But then SNK is the plane angle of the dihedral angle at the edge AD, one face of which passes through the point S, and the other through the point K. Let us denote this dihedral angle by $\angle SADK$. Thus, $\angle SNK$ is the plane angle of the dihedral angle $SADK$. Assuming that NM is the representation of the bisector of the angle SNK, we, at the same time, assume that the plane $APQD$ bisects the dihedral angle $SADK$, which satisfies the conditions of the problem. But assuming that NM is the representation of the bisector of the angle SNK, we spend one more parameter on the representation. Thus, all the five parameters have been spent. In addition, we have to take into consideration that the plane of the section divides the lateral surface of the pyramid into two equal parts. As we shall see later on, the assumption that the bisecting plane $APQD$ is the representation of the plane, which divides the lateral surface area of the pyramid into two equal parts, involves the assumption that $AD:SM = \sqrt{2}:1$. And this means that one more (the sixth) parameter is spent on the representation.

Thus, we shall solve this example on the representation for which $p = 6$. Passing over to all necessary computations, we note that since the pyramid is regular, the dihedral angles at the edges of the base are equal to one another. Therefore, it makes no difference which of these angles will be found. It is clear that it is more expedient to find the dihedral angle at the edge AD, since the cutting plane also passes through this edge. It has been already proved that $\angle SNK$ is the plane angle of the desired dihedral angle $SADK$. We set $\angle SNK = x$. At the edge AD two more dihedral angles are formed: the faces of one of them pass through the points S and M, while those of the other through the points K and M. The first of them is $\angle SADM$, and the second $\angle MADK$. The plane angle of the first of them is $\angle SNM$, and of the second $\angle MNK$. Since the cutting plane bisects the angle $SADK$, we have: $\angle SADM = \angle MADK$, and then $\angle SNM = \angle MNK$ as well, each of these angles being equal to $\frac{x}{2}$.

Let us now introduce an auxiliary parameter, setting the side of the base equal to a. It follows from the conditions of the problem that

$$S_{\triangle SAD}+2S_{\triangle SQD}+S_{\triangle SPQ}=2S_{\triangle DQC}+S_{BPQC}. \qquad (1)$$

Compute all the areas entering Equality (1). For brevity, set the length of the slant height of the lateral face equal to l.

(1) $S_{\triangle SAD}=\frac{1}{2}al$, $S_{\triangle SCD}=\frac{1}{2}al$, and $S_{\triangle SBC}=\frac{1}{2}al$.

(2) We have from the right triangle SON: $\cos x=\frac{a}{2l}$. Further, $\begin{cases} SM+MK=l, \\ \frac{SM}{MK}=\frac{SN}{NK}=\frac{l}{a}, \end{cases}$ whence $MK=\frac{al}{a+l}$ and $SM=\frac{l^2}{a+l}$.

We have from the triangle MNK (by the law of sines): $\frac{MK}{\sin\frac{x}{2}}=\frac{NK}{\sin\left(180°-\left(x+\frac{x}{2}\right)\right)}$, whence

$$\frac{l}{a+l}\sin\frac{3x}{2}=\sin\frac{x}{2}. \qquad (2)$$

(3) Construct QF, i.e. the altitude of the triangle DQC. Then $S_{\triangle DQC}=\frac{1}{2}aQF$. Construct SE, i.e. the slant height of the face SDC. Find QF from the proportion $\frac{QF}{SE}=\frac{QC}{SC}$. But $\frac{QC}{SC}=\frac{MK}{SK}$, that is, $\frac{QF}{SE}=\frac{MK}{SK}$, whence $QF=MK=\frac{al}{a+l}$ Thus, $S_{\triangle DQC}=\frac{a^2l}{2(a+l)}$.

(4) $S_{\triangle SQD}=S_{\triangle SCD}-S_{\triangle DQC}=\frac{1}{2}al-\frac{a^2l}{2(a+l)}=\frac{al^2}{2(a+l)}$.

(5) $S_{\triangle SPQ}=\frac{1}{2}SM\cdot PQ$. But $\frac{PQ}{BC}=\frac{SM}{SK}$, whence $PQ=\frac{al}{a+l}$. Thus, $S_{\triangle SPQ}=\frac{al^3}{2(a+l)^2}$.

(6) $S_{BPQC}=S_{\triangle SBC}-S_{\triangle SPQ}=\frac{1}{2}al-\frac{al^3}{2(a+l)^2}=\frac{a^2l(a+2l)}{2(a+l)^2}$.

(7) Substituting the found values of the areas into Equality (1), we find:

$$a=l\sqrt{2}. \qquad (3)$$

(Hence, there follows the relationship $AD:SN=\sqrt{2}:1$, which was mentioned when counting the parametric number of the representation.) Expressing the value of a from Equation (2), we get:

$a=\dfrac{l\sin\frac{3x}{2}}{\sin\frac{x}{2}}-l$. Substituting this value of a into Equality (3), we arrive at the equation: $\dfrac{\sin\frac{3x}{2}}{\sin\frac{x}{2}}-1=\sqrt{2}$, whence we get: $\sin\frac{3x}{2}-\sin\frac{x}{2}=\sqrt{2}\sin\frac{x}{2}$, or $2\sin\frac{x}{2}\cos x=\sqrt{2}\sin\frac{x}{2}$. Since $\sin\frac{x}{2}\neq 0$, from the last equation we have: $\cos x=\frac{\sqrt{2}}{2}$, whence $x=45°$.

PROBLEMS TO BE SOLVED WITHOUT ASSISTANCE

763. In a regular quadrangular prism, the diagonal is equal to d and is inclined at an angle equal to α to the plane of each lateral face. Find the lateral surface area of the prism.

764. The angles formed by the diagonal of the base of a rectangular parallelepiped with the side of the base and the diagonal of the parallelepiped are equal to α and β, respectively. Find the lateral surface area of the parallelepiped if its diagonal is d.

765. The altitude of a regular quadrangular prism is equal to H, and the angle between the diagonals drawn from one vertex of the base in two adjacent lateral faces is equal to α. Find the lateral surface area of the prism.

766. The altitude of a regular triangular prism is equal to H. A straight line passing through the centroid of the base and the midpoint of the side of the lower base forms an angle equal to α with the plane of the base. Find the total surface area of the prism.

767. The total surface area of a regular quadrangular pyramid is equal to S. The dihedral angle at the edge of the base equals α. Find the side of the base.

768. Find the total surface area of a regular quadrangular pyramid if its altitude is equal to H, and the area of each lateral face is equal to the area of the base.

769. The base of a pyramid is a right triangle whose legs are equal to 6 cm and 8 cm. The angle of inclination of each lateral face of the pyramid to the plane of the base is equal to 60°. Find the lateral surface area of the pyramid.

770. The lateral surface area of a regular quadrangular pyramid is equal to S, and the dihedral angle at each edge of the base is equal to α. Find the distance between the centroid of the base and the lateral face of the pyramid.

771. The total surface area of a regular triangular pyramid is equal to S, and the plane angle at the vertex of the pyramid is equal to α. Find the radius of the circle circumscribed about the base.

772. The base of a pyramid is a square whose side is a. Two lateral faces are perpendicular to the plane of the base, and each of the two other lateral faces forms an angle equal to α with it. Find the total surface area of the pyramid.

773. The base of a pyramid is a rectangle. Two adjacent lateral faces are perpendicular to the plane of the base, and two others form angles equal to α and β, respectively, with it. The altitude of the pyramid is equal to H. Find the lateral surface area of the pyramid.

774. The base of a pyramid is a triangle, the ratios of whose sides are 13:14:15, and each of the dihedral angles at the edges of the base is equal to 45°. Find the ratio of the total surface area of the pyramid to the area of its base.

775. The base of a right prism is an isosceles triangle in which the angle between the sides is equal to 2α. Drawn from the vertex of the upper base are two diagonals of the equal lateral faces. The angle between these diagonals is equal to 2β. Find the ratio of the lateral surface area of the prism to the area of its base.

776. The centroid of one face of a cube is joined to the vertices of the opposite face. Find the ratio of the total surface area of the pyramid thus obtained to the total surface area of the cube.

777. Drawn through a side of the lower base of a regular triangular prism and the midpoint of the lateral edge, not intersecting with this side, is a plane making an angle equal to α with the plane of the base. Find the ratio of the lateral surface area of the pyramid thus formed to the lateral surface area of the given prism.

778. The angle between the elements in the axial section of a cone is equal to 2α. Find the ratio of the lateral surface area of the cone to the area of its axial section.

779. The greatest angle between the elements of a cone is equal to 120°. Prove that the lateral surface area of this cone is equal to the lateral surface area of the cylinder having the same base and altitude.

780. The lateral surface area of a cone is a quarter of a circle rolled up to form a conical surface. Find the ratio of the total surface area of the cone to the area of its axial section.

781. The lateral surface area of a frustum of a cone is equal to the sum of the areas of its bases, and the radii of the bases are to each other as 1:3. Find the angle at which the generatrix is inclined to the plane of the base.

782. A regular triangle whose side is equal to a revolves about its axis, which is parallel to a side of the triangle and is passed through the vertex opposite to this side. Find the surface area of the solid of revolution thus generated.

783. A right triangle whose legs are equal to 5 cm and 12 cm revolves about its external axis, which is parallel to the larger leg and 3 cm distant from it. Find the surface area of the solid of revolution thus generated.

784. A rectangle whose sides are equal to a and b revolves about its axis, which is perpendicular to its diagonal and is passed through one of its end points. Find the surface area of the solid of revolution thus generated.

785. An isosceles triangle with base equal to a and base angle equal to α revolves about its axis passing through one of the end points of the base perpendicular to the base. Find the surface area of the solid of revolution thus generated.

786. In a right trapezoid circumscribed about a circle of radius R, the acute angle is equal to α. Find the surface area of the solid generated by revolving this trapezoid about the smaller of its parallel sides.

787. A rhombus with acute angle equal to α revolves about its side. Find the ratio of the surface area of the solid of revolution thus generated to the area of the rhombus.

788. The angle between the sides of a right triangle is equal to 60°. A straight line cuts off the sides of this triangle two line segments whose lengths are one-fourth of the length of the hypotenuse as measured from the vertex of this angle. Find the ratio of the area of the given triangle to the area of the surface generated by revolving this triangle about the given line.

789. An isosceles trapezoid with base angle of 60° revolves about the bisector of this angle. Find the ratio of the surface area of the solid of revolution to the area of the trapezoid if the altitude of the trapezoid is $\frac{1}{\sqrt{3}}$ of the half-sum of its bases.

790. A cutting plane parallel to the base of a regular triangular pyramid bisects its lateral surface. Find the ratio in which the altitude is divided by this section.

791. A cutting plane passed through the side AD of the base of a regular quadrangular pyramid $SABCD$ is perpendicular to the face SBC and divides this face into two equivalent figures. Find the total surface area of the pyramid if $AD = a$.

792. In a regular quadrangular pyramid, a plane passed through a side of the base bisects the lateral surface and the dihedral angle at an edge of the base. Find the dihedral angle at the lateral edge of the pyramid.

793. A cutting plane is passed through an edge of the base of a regular quadrangular pyramid and cuts off a triangle whose area is S_1 from the opposite face. Find the lateral surface area of the pyramid, which is separated by the cutting plane from the given pyramid, if the lateral surface area of the latter is equal to S_2.

794. The base of a pyramid is a rhombus whose side is equal to a, and the acute angle between its sides is equal to α. Each of the dihedral angles at the edges of the base is equal to φ. Find the lateral surface area of the pyramid.

795. The base of a pyramid is an isosceles trapezoid whose diagonal is equal to d, and the angle between this diagonal and the larger base of the trapezoid is equal to α. Each lateral face of the pyramid is inclined at an angle equal to φ to the plane of the base. Find the total surface area of the pyramid.

796. The length of each side of the base of a triangular prism is equal to a. One of the vertices of the upper base is projected in the centroid of the lower base. The lateral edges are inclined at angles, each of which is equal to α, to the plane of the base. Find the lateral surface area of the prism.

797. The base of a parallelepiped whose lateral edge is equal to b is a square with side a. One of the vertices of the upper base is equidistant from all the vertices of the lower base. Find the total surface area of the parallelepiped.

798. The base of a prism is a regular triangle whose side is equal to a. Each lateral edge of the prism is equal to b, and the angle between one of the lateral edges and two adjacent sides of the base is equal to 45°. Find the lateral surface area of the prism.

799. By how many times is the distance from a luminous point to the centre of a ball greater than the radius of the ball if the area of the illuminated part of the surface of the ball is half the shaded part?

SEC. 14. VOLUMES

Example 1. The base of a right parallelepiped is a parallelogram whose sides are equal to a and b, and the obtuse angle is equal to φ. Find the volume of the parallelepiped if the smaller diagonal of the parallelepiped is equal to the larger diagonal of the base (Fig. 148).

Solution. Let the figure $ABCDA_1B_1C_1D_1$ be the representation of the given parallelepiped. This is a complete representation, and its parametric number $p = 5$. Indeed, assuming AA_1 to be the representation of the line segment perpendicular to the plane of the base, we spend two parameters. Assuming that $AD:CD = a:b$, we spend one more parameter. Assuming, for instance, that $\angle ABC$ is the representation of the angle which is equal to φ in the original, we spend one more parameter. Assuming the angle ABC to be the representation of the obtuse angle, we shall take the line segments BD and AC as the representations of the smaller and larger diagonals of the parallelogram $ABCD$. Therefore, we shall assume that $B_1D:AC = 1:1$. Thus, we spend one more parameter.

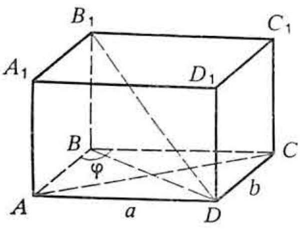


Fig. 148

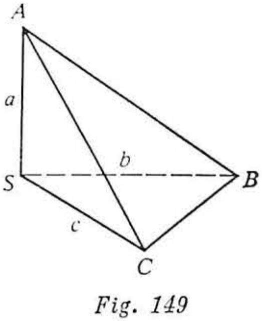


Fig. 149

Let us now find V, i.e. the volume of the parallelepiped. We have from the triangle ACD: $AC^2 = AD^2 + CD^2 - 2AD \cdot CD \cdot \cos\varphi$. But $AC = B_1D$. Thus, $B_1D^2 = a^2 + b^2 - 2ab\cos\varphi$. We get from the triangle ABD: $BD^2 = a^2 + b^2 - 2ab\cos(180° - \varphi) = a^2 + b^2 + 2ab\cos\varphi$. We have from the right triangle BB_1D: $BB_1^2 = B_1D^2 - BD^2$, that is, $BB_1^2 = (a^2 + b^2 - 2ab\cos\varphi) - (a^2 + b^2 + 2ab\cos\varphi) = -4ab\cos\varphi$, and, consequently, $BB_1 = 2\sqrt{-ab\cos\varphi}$. Since $S_{ABCD} = ab\sin\varphi$, we get: $V = 2ab\sin\varphi\sqrt{-ab\cos\varphi}$.

By the sense of the problem, φ is an obtuse angle. Therefore, $-1 < \cos\varphi < 0$, and then $(-\cos\varphi)$ is a positive number. Since $\sin\varphi > 0$, the found value of V is positive. Therefore, $V = 2ab\sin\varphi\sqrt{-ab\cos\varphi}$.

Example 2. The lateral faces of a triangular pyramid are pairwise perpendicular, their areas being equal to Q_1, Q_2, and Q_3. Find the volume of the pyramid.

Solution. Let the quadrilateral $SABC$ with its diagonals be the representation of the given pyramid (Fig. 149). This is a complete representation. Let us count its parametric number p. Since the lateral faces of the given pyramid are pairwise perpendicular, its lateral edges are also pairwise perpendicular. Assuming SA to be the representation of the edge of the pyramid perpendicular to its lateral edges SB and SC, we spend two parameters. Assuming SB to be the representation of the edge perpendicular to the edge SC, we spend one more parameter. Thus, having spent three parameters, we ensure the mutual perpendicularity of the lateral faces of the pyramid. Further, assuming that the triangles SAB and SBC are the representations of the lateral faces, the ratio of the areas of which is equal to $Q_1:Q_2$, that is, assuming that $\left(\frac{1}{2}SA \cdot SB\right):\left(\frac{1}{2}SB \cdot SC\right) = Q_1:Q_2$, or $SA:SC = Q_1:Q_2$, we spend one parameter. Analogously, assuming that the triangles SBC and SAC are the representations of the lateral faces, the ratio of the areas of which is equal to $Q_2:Q_3$, we also spend one parameter. Thus, for the constructed representation $p = 5$.

Let us now compute V, i.e. the volume of the pyramid. Note that regarding traditionally the triangle ABC as the base of the pyramid, we had, first of all, to compute the area of the triangle ABC and the altitude of the pyramid drawn from the vertex S to the plane ABC. In the example under consideration, however, it is possible to carry out all necessary computations more easily if we guess to "stand" the pyramid with a lateral face as base. Thus, noting that the edge

SA is perpendicular to the face SBC, we may take the triangle SBC as the base of the pyramid. Then $V = \frac{1}{3} SA \cdot S_{\Delta ABC} = \frac{1}{6} SA \cdot SB \cdot SC$. Setting, for brevity, $SA = a$, $SB = b$, and $SC = c$, we have: $V = \frac{1}{6}\, abc$. We find from the right triangles SAB, SBC, and SAC that $ab = 2Q_1$, $bc = 2Q_2$, and $ac = 2Q_3$. Multiplying the three equalities termwise, we find that $(abc)^2 = 8Q_1Q_2Q_3$, whence $abc = 2\sqrt{2Q_1Q_2Q_3}$. Thus, $V = \frac{1}{3}\sqrt{2Q_1Q_2Q_3}$.

Example 3. The lateral surface area of a cone, the radius of the base of which is R, is equal to the sum of the areas of the base and the axial section. Find the volume of the cone.

Solution. Let the ellipse ω together with a pair of tangents drawn to it from an outside point S be the representation of the given cone (Fig. 150). This is a complete representation. Assuming the ellipse ω to be the representation of the circle, we spend two parameters. Assuming the line segment SO to be the representation of the altitude of the cone, we spend two parameters. Let us construct AB, that is, the representation of the diameter of the circle, as well as SA and SB, that is, the representations of the elements of the cone. No parameter is spent on these constructions. Finally, assuming that we are given the representation of such a cone in which $S_{\text{lat}} = S_{\text{base}} + S_{\Delta SAB}$, we spend one more parameter. (Indeed, it follows from this equality that the ratio $SO:AO$ is determined identically.) Thus, for the constructed representation $p = 5$.

Now, we are going to determine V, i.e. the volume of the cone: $V = \frac{1}{3} S_{\text{base}} \cdot SO$, where $S_{\text{base}} = \pi R^2$. Let us set, for brevity, that $SO = x$. Thus, to compute V, it is necessary to find x. Since $S_{\text{lat}} = \pi R \cdot SA$, $S_{\text{base}} = \pi R^2$, and $S_{\Delta SAB} = Rx$, we set up the equation: $\pi R \cdot SA = \pi R^2 + Rx$, or $\pi SA = \pi R + x$. But we have from the right triangle SAO: $SA = \sqrt{x^2 + R^2}$. Thus, we get: $\pi\sqrt{x^2 + R^2} = \pi R + x$. Squaring both sides of this equation and simplifying the result, we get: $(\pi^2 - 1)\, x^2 - 2\pi R x = 0$, whence $x_1 = \frac{2\pi R}{\pi^2 - 1}$ and $x_2 = 0$. It is clear that the second solution does not satisfy the conditions of the problem. Consequently, $V = \frac{2\pi^2 R^3}{3(\pi^2 - 1)}$.

Example 4. A rectangle with sides a and b revolves about the axis passing through its vertex parallel to the diagonal not passing through this vertex. Find the volume of the solid of revolution thus generated.

Solution. The same as in Example 4 of the preceding section, we confine ourselves not to the solid of revolution, but only to the

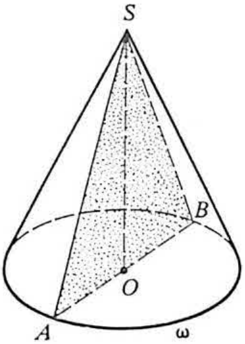


Fig. 150

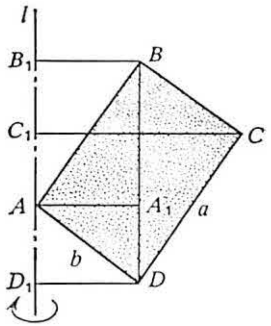


Fig. 151

figure obtained as a section of the solid of revolution by the half-plane bounded by the axis of revolution. Such a section is a rectangle. The rectangle $ABCD$ with the diagonal BD and the straight line l parallel to the diagonal BD and passing through the point A is the representation of this section (Fig. 151). (In this case, on the representation we construct a figure similar to the original, that is, no parameter is spent here.)

Thus, let $AB=a$ and $AD=b$. Let us compute V_{ABCD}, i.e. the volume of the solid of revolution. From the points B, C, and D we drop the perpendiculars BB_1, CC_1, and DD_1 to the straight line l, and from the point A the perpendicular AA_1 to the straight line BD. Then $V_{ABCD}=(V_1+V_2)-(V_3+V_4)$, where $V_1=V_{C_1B_1BC}=\frac{1}{3}\pi B_1C_1(CC_1^2+CC_1\cdot BB_1+BB_1^2)$, $V_2=V_{D_1C_1CD}=\frac{1}{3}\pi C_1D_1(CC_1^2+CC_1\cdot DD_1+DD_1^2)$, $V_3=V_{AB_1B}=\frac{1}{3}\pi AB_1\cdot BB_1^2$, and $V_4=V_{AD_1D}=\frac{1}{3}\pi AD_1\cdot DD_1^2$. But since $l \parallel BD$, we have: $BB_1=DD_1=AA_1$. We find from the right triangle ABD that $BD=\sqrt{a^2+b^2}$ and $AA_1=\frac{ab}{\sqrt{a^2+b^2}}$. Further, we find that $CC_1=2AA_1=\frac{2ab}{\sqrt{a^2+b^2}}$. Then $V_1+V_2=\frac{1}{3}\pi(CC_1^2+CC_1\cdot AA_1+AA_1^2)(B_1C_1+C_1D_1)=\frac{1}{3}\pi\left(\frac{4a^2b^2}{a^2+b^2}+\frac{2ab}{\sqrt{a^2+b^2}}\cdot\frac{ab}{\sqrt{a^2+b^2}}+\frac{a^2b^2}{a^2+b^2}\right)\sqrt{a^2+b^2}=\frac{1}{3}\pi\frac{7a^2b^2}{\sqrt{a^2+b^2}}$. Analogously, $V_3+V_4=\frac{1}{3}\pi AA_1^2(AB+AD)=\frac{1}{3}\pi\frac{a^2b^2}{\sqrt{a^2+b^2}}$. Thus, $V_{ABCD}=\frac{1}{3}\pi\frac{6a^2b^2}{\sqrt{a^2+b^2}}=\frac{2\pi a^2b^2}{\sqrt{a^2+b^2}}$.

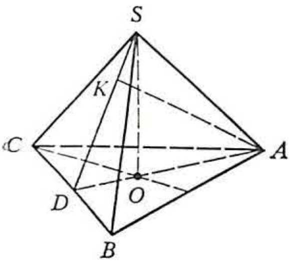


Fig. 152

Remark. The desired volume of the solid of revolution can be computed more easily if we take advantage of the second Guldin's theorem, according to which $V = S2\pi R$, where S is the area of the figure revolving about the axis, and R the distance from the centroid of this figure to the axis of revolution. In the example under consideration, $S = ab$ and $R = AA_1 = \frac{ab}{\sqrt{a^2+b^2}}$.]

Example 5. Each side of the base of a regular triangular pyramid is equal to a, and the altitude drawn from one of the vertices of the base to the opposite lateral face is equal to b. Find the volume of the pyramid.

Solution. Let the quadrilateral $SABC$ with its diagonals be the representation of the given pyramid (Fig. 152). This is a complete representation. Assuming the triangle ABC to be the representation of the regular triangle, we spend two parameters. Assuming SO to be the representation of the altitude of the pyramid, we also spend two parameters. We then construct SD, i.e. a median of the triangle SBC, that is, the representation of the slant height of a lateral face of the pyramid. Let AK be the representation of the perpendicular drawn from the vertex A of the given pyramid to the slant height of the opposite lateral face. It is not difficult to show that assuming the line segment AK to be the representation of the altitude drawn from the vertex of the base to the opposite lateral face, we spend one more parameter. Thus, all the five parameters have been spent on the representation. However, the condition, according to which $A_0B_0 = a$ and $A_0K_0 = b$ in the original, is not taken into account. The fact that the indicated data are given in letters offers us some freedom. Assuming that on the constructed representation $AB:AK = a:b$, we spend one more parameter. But then the representation becomes metrically overdetermined. Nevertheless, with a certain relationship between a and b fulfilled (we shall get this relationship below), the constructed representation is true.

Taking an arbitrary point K on the slant height SD, let us show that it is impossible to assume that AK is the representation of the line segment A_0K_0, which is perpendicular to the line segment S_0D_0 in the original. Since, by the sense of the problem, the line segment A_0K_0 exists, we can compute from the triangle $S_0A_0D_0$, where $A_0K_0 = b$ and $A_0D_0 = \frac{a\sqrt{3}}{2}$, that $D_0K_0:S_0K_0 = \frac{3a^2-4b^2}{4b^2-2a^2}$. Choosing various (permissible) values of a and b or various values of the ratio $a:b$, we shall get various values of the ratio $D_0K_0:S_0K_0$, that is, various positions of the point K_0.

Thus, given, for instance, $\begin{cases} a=10, \\ b=8, \end{cases}$ we would get: $\frac{3a^2-4b^2}{4b^2-2a^2}=\frac{11}{14}$. For such concrete values of a and b, as we see, the position of the point K_0 is determined identically since $D_0K_0 : S_0K_0 = 11:14$. This means that it is already impossible to represent the point K_0 arbitrarily.

Thus, let $SABC$ be the given regular pyramid in which $AB=a$ and $AK=b$. Find V, i.e. the volume of this pyramid. Since $V=\frac{1}{3}S_{\triangle ABC}\cdot SO$, where $S_{\triangle ABC}=\frac{a^2\sqrt{3}}{4}$, to determine the volume it is sufficient to find the altitude SO. We get from the similarity of the right triangles SOD and ADK: $\frac{SO}{AK}=\frac{OD}{DK}$, whence $SO=\frac{AK\cdot OD}{DK}$, where $AK=b$, $OD=\frac{1}{3}AD=\frac{a\sqrt{3}}{6}$, and $DK=\sqrt{AD^2-AK^2}=\frac{\sqrt{3a^2-4b^2}}{2}$. Thus, $SO=\frac{ab\sqrt{3}}{\sqrt{3a^2-4b^2}}$. Then $V=\frac{a^3b}{12\sqrt{3a^2-4b^2}}$.

By hypothesis, AK is the line segment drawn to the face SBC (just to the face, but not to the plane of the face!). It is clear that if the given pyramid is so low that $\angle ASD=90°$, then the points K and S coincide, and, hence, the line segments AK and SA coincide. If the pyramid is still lower, then the line segment drawn to the face SBC does not exist at all. Setting $\angle ASD=90°$, we find from the right triangle ASD, where $OD=\frac{a\sqrt{3}}{6}$ and $AO=\frac{a\sqrt{3}}{3}$ that $SO=\sqrt{\frac{a\sqrt{3}}{6}\,\frac{a\sqrt{3}}{3}}=\frac{a\sqrt{6}}{6}$. Thus, for the line segment AK to exist, it is necessary that the values of a and b satisfy the inequality: $\frac{ab\sqrt{3}}{3\sqrt{3a^2-4b^2}}\geqslant\frac{a\sqrt{6}}{6}$ or the equivalent system of inequalities: $\frac{a\sqrt{2}}{2}\leqslant b\leqslant\frac{a\sqrt{3}}{2}$. At the same time, it is clear that, by hypothesis, $AK<AD$, since AK is perpendicular to the face SBC, while AD is a line inclined to this face. Then $b<\frac{a\sqrt{3}}{2}$. This relationship between a and b has been already revealed above. Thus, if $\frac{a\sqrt{2}}{2}\leqslant b<\frac{a\sqrt{3}}{2}$, then the pyramid satisfying the conditions of the problem exists. Thus, $V=\frac{a^3b}{12\sqrt{3a^2-4b^2}}$, where $\frac{a\sqrt{2}}{2}\leqslant b<\frac{a\sqrt{3}}{2}$.

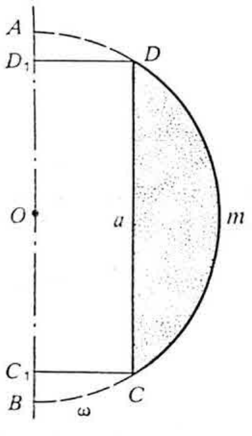


Fig. 153

Example 6. Prove that the volume of the solid generated by revolving a segment of a circle subtended by a chord whose length is equal to a about the diameter of this circle, which is parallel to the chord, is independent of the radius of the circle.

Solution. Let us confine ourselves to representing not the solid of revolution, but only to the figure obtained as a section of the solid of revolution by the half-plane bounded by the axis of revolution.

Let the segment CmD of the circle ω be the representation of this section, and the diameter AB of the circle be the representation of the axis of revolution (Fig. 153). The entire representation is a plane figure, and, therefore, the completeness of representation and counting of parameters are out of question here: on the representation we construct a figure similar to the original.

Thus, let the chord CD be equal to a, and let this chord be parallel to the axis of revolution AB. Let us prove that V, i.e. the volume of the solid of revolution generated by revolving the segment CmD about the axis AB, is independent of OA. To this end, we construct $CC_1 \perp AB$ and $DD_1 \perp AB$. Then $V = V_1 - V_2 - 2V_3$, where V_1 is the volume of the ball whose radius is equal to OA, V_2 the volume of the cylinder whose base is a circle of radius DD_1 and whose altitude is equal to C_1D_1, and V_3 the volume of the segment of the ball whose base is a circle of radius DD_1 and whose altitude is equal to $OA - OD_1$. We set, for brevity, that $OA = R$. Then $V_1 = \frac{4}{3}\pi R^3$, $V_2 = \pi DD_1^2 C_1D_1 = \pi\left(\frac{\sqrt{4R^2-a^2}}{2}\right)^2 a$, and $V_3 = \pi AD_1^2\left(OA - \frac{1}{3}AD_1\right) = \pi\left(R - \frac{a}{2}\right)^2 \times \left(R - \frac{1}{3}\left(R - \frac{a}{2}\right)\right) = \pi\left(R - \frac{a}{2}\right)^2\left(\frac{2}{3}R + \frac{a}{6}\right)$. Thus, $V = \frac{4}{3}\pi R^3 - \frac{4\pi R^2 a - \pi a^3}{4} - \frac{2\pi}{3}\left(R^2 - aR + \frac{a^2}{4}\right)\left(2R + \frac{a}{2}\right)$, and, after simplifications, we get: $V = \frac{\pi a^3}{6}$. Since the expression $\frac{\pi a^3}{6}$ does not contain R, the required statement has been proved.

Example 7. A triangular pyramid is cut by a plane into two polyhedrons. Find the ratio of the volumes of the obtained polyhedrons if it is known that the cutting plane divides the edges meeting at one vertex of the pyramid in the ratio 1:2, 1:2, 2:1 as measured from this vertex.

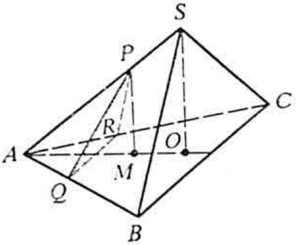


Fig. 154

Solution. Let the quadrilateral $SABC$ with its diagonals be the representation of the given pyramid (Fig. 154), and let the triangle PQR be the representation of the given section. This is a complete representation, and not a single parameter is spent on it.

Let V be the volume of the pyramid $SABC$, and V_1 the volume of the pyramid $PAQR$. We have: $AQ:QB=1:2$, $AR:RC=1:2$, and $AP:PS=2:1$. Find the desired ratio $\frac{V-V_1}{V_1}$. Taking an arbitrary point O in the plane ABC and joining it to the vertex S, we shall assume that SO is the representation of the altitude of the pyramid $SABC$ (two parameters are spent here). We then construct AO and $PM \parallel SO$. Then the point M lies on the straight line AO. To make the calculations look simpler, we set $AB=a$, $AC=b$, $AS=c$, $SO=H$, and $\angle BAC=\alpha$. Then $AQ=\frac{1}{3}a$, $AR=\frac{1}{3}b$, and $AP=\frac{2}{3}c$, and we get from the similarity of the triangles APM and ASO: $PM=\frac{2}{3}H$. Now, compute V and V_1. We have: $V=\frac{1}{3}S_{\Delta ABC}H=\frac{1}{6}abH\sin\alpha$ and $V_1=\frac{1}{3}S_{\Delta AQR}\frac{2}{3}H=\frac{1}{81}abH\sin\alpha$. Then $V=V_1-\frac{25}{162}abH\sin\alpha$. Thus, $\frac{V-V_1}{V_1}=$

$$\frac{\frac{25}{162}abH\sin\alpha}{\frac{1}{81}abH\sin\alpha}=\frac{25}{2}.$$

PROBLEMS TO BE SOLVED WITHOUT ASSISTANCE

800. The base of a parallelepiped is a rhombus $ABCD$ whose side is equal to a, and the acute angle is equal to 60°. Find the volume of the parallelepiped if the lateral edge is equal to a, and $\angle A_1AB=\angle A_1AD=45°$.

801. Each edge of a parallelepiped is equal to a. Each of the three plane angles at one vertex of the parallelepiped is equal to 2α. Find the volume of the parallelepiped.

802. The edges of a parallelepiped equal to a and b are mutually perpendicular, and the edge whose length is c forms an angle equal to α with each of the first two edges. Find the volume of the parallelepiped.

803. The area of one of the lateral faces of a triangular prism is equal to m^2. Find the volume of the prism if the distance from the opposite edge to the plane of this face is equal to $2a$.

804. The base of a right triangular prism is an isosceles triangle whose equal sides have a length a and form an angle equal to α. The diagonal of the face opposite to the angle makes an angle equal to φ with another lateral face. Find the volume of the prism.

805. A regular quadrangular prism, the side of the base of which is equal to a, is truncated so that each of its two adjacent lateral edges is equal to b, and each of the other two edges is equal to c. Find the volume of this truncated prism.

806. A cutting plane is passed through a side of the base of a regular triangular prism at an angle α to the plane of the base and cuts off a pyramid whose volume is equal to V. Find the area of the section.

807. In a regular quadrangular pyramid, the plane passing through a side of the base and the midline of the opposite lateral face makes an angle of 60° with the plane of the base. Find the volume of the pyramid if the side of the base is equal to a.

808. The base of a pyramid is an isosceles triangle with the sides equal to a (each) and the vertex angle α. The lateral faces of the pyramid form angles each of which is equal to 45° with the plane of the base. Find the volume of the pyramid.

809. In a triangular pyramid, all the lateral edges and two edges of the base are equal to a (each). The angle between the equal sides of the base is equal to 2α. Find the volume of the pyramid.

810. The area of the diagonal section of a regular quadrangular pyramid is equal to S. Each lateral edge makes an angle equal to α with the plane of the base. Find the volume of the pyramid.

811. The altitude of a regular triangular pyramid is equal to H. The dihedral angle between the lateral faces is equal to φ. Find the volume of the pyramid.

812. Each lateral edge of a regular quadrangular pyramid is equal to b. The dihedral angle between two adjacent lateral faces equals φ. Find the volume of the pyramid.

813. Find the surface area of a regular triangular pyramid whose volume is equal to V, and the angle between each lateral face and the plane of the base is equal to α.

814. The base of a pyramid is a right triangle whose hypotenuse is equal to c, and the acute angle is equal to α. Each lateral edge of the pyramid is inclined at an angle β to the plane of the base. Find the volume of the pyramid.

815. The perpendicular dropped from the centroid of the base of a regular triangular pyramid to its lateral edge is equal to l. Find the volume of the pyramid if the dihedral angle between the lateral face and the plane of the base of the pyramid is equal to α.

816. The perpendicular dropped from the centroid of the base of a regular triangular pyramid to its lateral face is equal to l. Find the volume of the pyramid if the angle between the lateral edge and the plane of the base is equal to β.

817. In a triangular pyramid, each of the lateral edges of which is equal to b, one of the plane angles at the vertex is equal to 90°, and each of the two others equals 60°. Find the volume of the pyramid.

818. In a triangular pyramid, the areas of two mutually perpendicular faces are equal to P and Q, the length of their common edge being equal to b. Find the volume of the pyramid.

819. The altitude of a pyramid whose base is a square lies outside the pyramid and is equal to H. Two opposite lateral faces are isosceles triangles making angles equal to α and β with the plane of the base. Find the volume of the pyramid.

820. A cutting plane drawn through the side AC of the base of a regular triangular pyramid $SABC$ perpendicular to the edge SB cuts off the pyramid $DABC$ whose volume is $\frac{1}{1.5}$ the volume of the pyramid $SABC$. Find the area of the lateral surface of the pyramid $SABC$ if $AC = a$.

821. The base of a quadrangular pyramid $SABCD$ is a parallelogram $ABCD$. The lateral faces SAB and SBC are perpendicular to the plane of the base. Drawn through the midpoints of the edges AD and CD is a cutting plane parallel to the edge SB. Find the ratio of the volumes of the solids of revolution thus generated.

822. The base of a pyramid is a rectangle. Two lateral faces are perpendicular

to the plane of the base, and two others form angles equal to α and β with it. Find the area of the base of the pyramid if its volume is equal to V.

823. The base of a pyramid is a trapezoid, the lateral sides and the smaller base of which are equal to a (each), and the angle between the lateral side and the base is equal to α. Each lateral edge is inclined at an angle equal to β to the plane of the base. Find the volume of the pyramid.

824. The base of a pyramid is an isosceles trapezoid whose acute angle is equal to α, and the area is equal to S. Each lateral face of the pyramid forms an angle equal to β with the plane of the base. Find the volume of the pyramid.

825. A triangle ABC in which $AC = b$, $AB = c$, and $\angle BAC = \alpha$ revolves about the axis, which passes through the vertex A outside the triangle and makes equal angles with the sides AC and AB. Find the volume of the solid of revolution thus generated.

826. An isosceles trapezoid whose acute angle is equal to 45°, and whose lateral side is equal to the smaller base revolves about its lateral side. Find the volume of the solid of revolution thus generated if the lateral side of the trapezoid is equal to b.

827. A right triangle revolves about the axis passing through the vertex of the right angle parallel to the hypotenuse. Find the volume of the solid of revolution thus generated if it is known that the area of the triangle is equal to S, and the perpendicular dropped from the vertex of the right angle to the hypotenuse is equal to half the length of one of the legs.

828. Find the ratio of the volumes of the solids generated by revolving a triangle about its base and about a straight line parallel to the base and passing through the vertex of the triangle.

829. Prove that the volumes of the solids generated by revolving a parallelogram about its adjacent sides are inversely proportional to these sides.

830. A triangle with sides, the ratio of which is equal to $a:b:c$, revolves first about one of its sides, then about the other, and, finally, about the third. Find the ratio of the volumes of the solids of revolution thus generated.

831. When a right triangle revolves about its legs and then about the hypotenuse, solids of revolution are generated whose volumes are equal to V_1, V_2, and V_3, respectively. Prove that $\frac{1}{V_3^2} = \frac{1}{V_2^2} + \frac{1}{V_1^2}$.

832. The sides of the base of a frustum of a regular quadrangular pyramid are equal to a and b $(a > b)$. The angle formed by the plane of a lateral face and the plane of the base is equal to α. Find the volume of the pyramid.

833. Taken on two skew lines are line segments whose lengths are equal to a and b. Prove that the volume of the parallelepiped whose edges are represented by these segments is independent of the position of the segments on the given lines.

834. The radius of the base of a cone is equal to R. Two mutually perpendicular elements divide the area of the lateral surface of the cone in the ratio 1:2. Find the volume of the cone.

835. From a circle whose radius is equal to R a sector with central angle α is cut out. The sector is rolled up into a conical funnel. Find the volume of the funnel.

836. The base of a pyramid is a regular triangle whose side is equal to a. The perpendicular dropped from the midpoint of the smaller lateral edge to the plane of the opposite face is equal to $\frac{a}{4}$. The foot of the altitude lying outside the pyramid is equidistant from two vertices of the base (triangle), the distance from the third vertex being half the distance from the first two. Find the volume of the pyramid.

837. Drawn in a cylinder parallel to its axis and at a distance a from it is a plane cutting off an arc α from the circle of the base. The area of the section is equal to S. Find the volume of the cylinder.

838. The area of the axial section of a sector of a sphere is one-third of the area of the great circle of the sphere. Find the ratio of the volume of this sector to the volume of the sphere.

839. In a regular quadrangular pyramid, the area of the section parallel to the base is one-third of the area of the base. Find the ratio in which the volume of the pyramid is divided by this section.

840. Given a cube $ABCDA_1B_1C_1D_1$, where M is the centroid of the face AA_1B_1B, N is the midpoint of the edge CC_1, K lies on the edge DC, and $DK = \frac{1}{4}DC$. The plane drawn through the points M, N, and K divides the cube into two polyhedrons. Find the ratio of their volumes.

841. Given a cube $ABCDA_1B_1C_1D_1$, where M is the midpoint of the edge AA_1, and N is the midpoint of the edge A_1B_1. The plane drawn through the points M, N, and C divides the cube into two polyhedrons. Find the ratio of their volumes.

842. A cutting plane is passed through the points P, Q, and R lying on the extensions of the respective edges AB, AA_1, and AD of the cube $ABCDA_1B_1C_1D_1$, where $AP{:}BP = AQ{:}A_1Q = AR{:}DR = 5{:}3$. Find the ratio of the volumes thus obtained.

843. Given a triangular prism $ABCA_1B_1C_1$. Find the ratio in which the volume of the prism is divided by the cutting plane intersecting the edges A_1B_1, B_1C_1, and BC at the respective points M, N, and K if $B_1M{:}A_1B_1 = 1{:}2$, $B_1N{:}B_1C_1 = 2{:}3$, and $BK{:}CB = 1{:}3$.

844. Given a right triangular prism $ABCA_1B_1C_1$ in which $AC{:}AA_1 = 3{:}4$. Find the ratio in which the volume of the prism is divided by the plane drawn through the vertex A and intersecting the lateral edges BB_1 and CC_1 at the respective points M and N if $BM = MB_1$, and AN is the bisector of the angle CAC_1.

845. A cutting plane is passed through the point M lying on the extension of the edge AB of a regular triangular prism $ABCA_1B_1C_1$, the vertex B_1, and the midpoint of the edge AC. Find the ratio of the volumes thus obtained if $AM{:}BM = 2{:}1$.

846. A cutting plane is passed through the points M, N, and P of the respective edges A_1B_1, B_1C_1, and BC of the triangular prism $ABCA_1B_1C_1$, where $A_1M = MB_1$, $B_1N{:}NC_1 = 2{:}1$, and $BP{:}PC = 1{:}2$. Find the ratio in which the cutting plane divides the volume of the prism.

847. A cutting plane is passed through the points K, L, and M of the respective edges SA, SB, and SC of a triangular pyramid, where $SK{:}KA = SL{:}LB = 2{:}1$, and the median SN of the face SBC is bisected by this plane. Find the ratio in which the cutting plane divides the volume of the pyramid.

848. A cutting plane is passed through the vertex A of the base of a triangular pyramid $SABC$, the point D, i.e. the midpoint of the median SK of the face SAB, and the point E of the median SL of the face SAC such that $SE{:}EL = 1{:}2$. Find the ratio in which the cutting plane divides the volume of the pyramid.

849. Given a regular quadrangular pyramid $SABCD$. Find the ratio in which the volume of the pyramid is divided by the plane drawn through the points A and B and the midpoint of the edge SC.

850. A plane passing through one of the edges of a regular tetrahedron divides its volume in the ratio 3:5. Find the tangents of the angles into which the dihedral angle of the tetrahedron is divided by this plane.

851. Drawn through each edge of a tetrahedron is a plane parallel to the opposite edge. Find the ratio of the volume of the parallelepiped thus obtained to the volume of the tetrahedron.

852. Given a cube $ABCDA_1B_1C_1D_1$, where E is the midpoint of the edge DC, and F is the midpoint of the edge BB_1. The volume of the pyramid $AFED_1$ is what part of the volume of the cube?

SEC. 15. COMBINATIONS OF POLYHEDRONS AND CIRCULAR SOLIDS

Prior to considering the examples given below, we should like to note that when solving problems involving a combination of geometrical figures, we sometimes have to simplify the relevant drawing, since it turns out to be too complicated. In some cases, it is sufficient to have only the representation of the component figures (as in most problems involving a combination of circular solids), while in other cases, it is necessary to have only the representation of one of the figures of a combination. Sometimes, we have to represent one of the figures completely and the other partially. When solving some problems, it turns out to be expedient to take advantage of a pair of orthogonal projections of the combined figures.

Example 1. A ball Ω touches the base of a regular triangular pyramid $SABC$ at the point B and its lateral edge SA. Find the radius of the ball if $AB = a$ and $SA = b$ (Fig. 155).

Solution. Let the quadrilateral $SABC$ with its diagonals be the representation of the given pyramid. This is a complete, metrically determined representation (make this sure independently). It is difficult to construct the representation of the ball, since its radius is unknown (as if we have a "vicious" circle: to construct the representation of the ball, it is necessary to know its radius, while to find the radius, it is desirable to have the representation of the ball). Let us try to solve the problem by means of the representation in which the centre of the ball is constructed, while the ball itself is not. We construct the centre of the ball. Note, first of all, that if the centre of the ball is represented by the point P, then the distance from the point P to the point B, at which the ball Ω touches the plane of the base, will be equal to the radius of this ball. Thus, to compute the radius of the ball, there is no need to have its representation. We construct the point P proceeding from the following considerations.

Since the ball Ω touches the plane ABC at the point B, the point P lies on the perpendicular to the plane ABC erected at the point B. The altitude SO of the pyramid is already represented in the drawing. Through the point B we draw a straight line m parallel to SO. Then, since SO is perpendicular to the plane ABC, the straight line m will also be perpendicular to the plane ABC. Thus, the point P lies on the straight line m, and the line segment PB is the radius of the ball Ω.

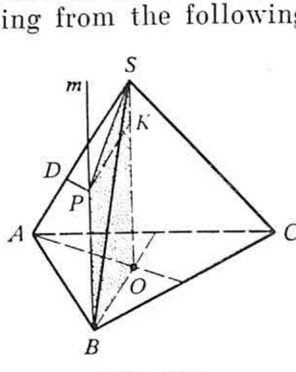


Fig. 155

If the ball touches the edge SA at some point D, then it is clear that $AB = AD$ (as the lengths of the line segments of tangents drawn to the ball from one point). But $AB = a$, and, consequently, to construct the point D, we have to lay off from the point A on the line segment SA such a line segment AD that $AD = a$. Taking advantage of the fact that the lengths of the edges AB and SA are given in general form, we take arbitrarily the point D on the straight line SA and assume that $AD = AB$. (If a and b are given by concrete numerical values, then, of course, it is impossible to take the point D arbitrarily. For instance, if $a = 15$ and $b = 20$, then $AD:SA = 15:20$, or $AD:SA = 3:4$, whence it is clear how to construct the point D.)

We then construct the line segment PD. Since the edge SA touches the ball Ω, we have: $PD \perp SA$ and PD is the radius of the ball Ω, i.e. $PD = PB$. To compute the length of the line segment PD, we carry out the following additional constructions:

(1) Join the point S to the point P.

(2) Draw OB, i.e. a median of the triangle ABC.

(3) Draw PK parallel to OB.

Setting $PB=x$, we set up an equation. Since $SA=b$ and $AD=a$, we have: $SD=b-a$. We find from the right triangle SDP that $SP^2=(b-a)^2+x^2$. We find from the right triangle SBO that $BO=\frac{a\sqrt{3}}{3}$ and $SO=\sqrt{b^2-\frac{a^2}{3}}$. Since $m \parallel SO$ and $PK \parallel OB$, we have: $OB=PK=\frac{a\sqrt{3}}{3}$ and $PB=KO=x$. Then $SK=\sqrt{b^2-\frac{a^2}{3}}-x$. Now, we find from the right triangle SPK that $SP^2=SK^2+PK^2$, or $(b-a)^2+x^2=\left(\sqrt{b^2-\frac{a^2}{3}}-x\right)^2+\left(\frac{a\sqrt{3}}{3}\right)^2$. Solving this equation, we get: $x=\frac{\sqrt{3}\,a\,(2b-a)}{2\sqrt{3b^2-a^2}}$. As was noted above, $AD=AB$, that is, $AD=a$. But $SA>AD$. Thus, according to the sense of the problem, $b>a$. Thus, $PB=\frac{\sqrt{3}\,a\,(2b-a)}{2\sqrt{3b^2-a^2}}$, where $b>a$.

Remark. We draw the reader's attention to the fact that the inequality $b > a$ is obtained following the sense of the problem, but not from the formula derived for the desired quantity. From the formula obtained for PB we would find that a and b must satisfy the inequalities of the system $\begin{cases} 2b-a>0, \\ 3b^2-a^2>0 \end{cases}$ (the inequalities $a>0$ and $b>0$ are omitted as obvious ones). From this system we would get: $b>\frac{a\sqrt{3}}{3}$.

However, this relationship turns out to be sufficient only for the existence of the pyramid itself: in the right triangle SAO the hypotenuse is greater than either of its legs. But for the ball touching SA to exist, the relationship $b > \frac{a\sqrt{3}}{3}$ is insufficient. Indeed, let $b = 0.8a$ $\left(\text{in this case, } b > \frac{a\sqrt{3}}{3}\right)$. As was mentioned above, $AD = a$. Thus, AD will be greater than SA, that is, the ball will touch not the line segment SA, but its extension.

Example 2. Given a cube $ABCDA_1B_1C_1D_1$ whose edge is equal to a. A sphere Ω is passed through the vertices A and C and the midpoints of the edges B_1C_1 and C_1D_1. Find the radius of this sphere (Fig. 156).

Solution. Let the figure $ABCDA_1B_1C_1D_1$ be the representation of the given cube, and F_1 and E_1 be the midpoints of its edges B_1C_1 and C_1D_1, respectively. This is a complete, metrically determined representation (make this sure independently). As in the preceding example, it is very difficult to construct the representation of the given sphere, since the length of the radius of the sphere is unknown. But the representation of the sphere itself is not so necessary: in fact, if the centre of this sphere and one of its points are shown on the representation (here, four such points are already shown), then, probably, the length of the radius can be computed.

Let us, first of all, look for the point M, i.e. the centre of the sphere Ω. Since the points A and C belong to the sphere Ω, the point M belongs to the locus of points equidistant from the points A and C. It is easy to guess that this locus is represented by the diagonal plane BB_1D_1D. Analogously, the point M belongs to the diagonal plane AA_1C_1C. Thus, the point M belongs to the straight line OO_1 along which the planes ACC_1 and BDD_1 intersect. Finally, the point M belongs to the plane α (for the sake of brevity, it is not shown in the figure), which cuts the line segment CE_1 at the point L, i.e. the midpoint of the line segment CE_1, the plane α being perpendicular to CE_1. Since the planes α and DCC_1 have a common point L, these planes intersect along the straight line l passing through the point L, and since $CE_1 \perp \alpha$, we have: $CE_1 \perp l$. We now construct the point E, i.e. the midpoint of the line segment CD, and then the line segment EE_1. Since the representation on which the additional constructions are carried out is metrically determined, it is impossible to draw the straight line l through the point L arbitrarily and to say that l is perpendicular to CE_1. At the same time, the straight line l intersects the line segment EE_1. Let us denote the point of their intersection by K. Then KLE_1 will be a right triangle similar to the triangle

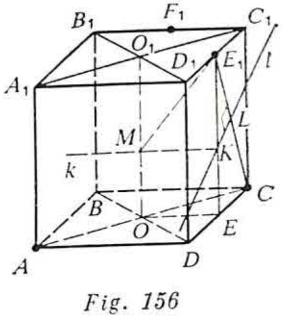


Fig. 156

CEE_1. The similarity of these triangles implies that $KE_1:CE_1 = LE_1:EE_1$, and since $EE_1 = a$ and $CE = \frac{a}{2}$, we have: $CE_1 = \frac{a\sqrt{5}}{2}$ and $LE_1 = \frac{a\sqrt{5}}{4}$. Thus, we find that if l is perpendicular to CE_1, then $KE_1 = \frac{5}{8}a$. Using this equality, we construct the point K. We then construct OE and draw the straight line k parallel to OE through the point K. The straight line k lies in the plane OEK and intersects OO_1. Let us denote the point of their intersection by M. Then, since MK is parallel to OE and OE is perpendicular to the plane DCC_1, MK is perpendicular to the plane DCC_1.

Hence, $CE_1 \perp LK$ and $CE_1 \perp MK$. Then CE_1 is perpendicular to the plane LKM. In other words, the plane LKM coincides with the plane α. Thus, the point M lies in the plane α, and, therefore, the point M is the centre of the sphere Ω.

Let us find ME_1, i.e. the radius of the sphere Ω. We find from the right triangle MKE_1 that $ME_1 = \sqrt{MK^2 + KE_1^2} = \frac{a\sqrt{41}}{8}$.

(We could find MC, i.e. the radius of the sphere Ω, from the right triangle MCO, where $OM = \frac{3}{8}a$ and $OC = \frac{a\sqrt{2}}{2}$.)

Example 3. A trihedral angle is formed by the planes α, β, and γ, where $\alpha \perp \gamma$, $\beta \perp \gamma$, and $\angle\alpha\beta = 2\varphi$. A sphere Σ touches the plane γ at the point B and intersects the planes α and β along the circles ω_1 and ω_2 whose radii are equal to r. The distance from the point O, i.e. the centre of the sphere, to the point A, i.e. the vertex of the trihedral angle, is equal to l. Find the radius of the sphere.

Solution. To construct the three-dimensional representation of the combination of the sphere Σ and the trihedral angle $\alpha\beta\gamma$ is a very difficult thing. For this purpose, we are going to apply the method of orthogonal projection on a pair of mutually perpendicular projection planes H and V, which will be chosen in the following way. Since the sphere Σ touches the plane γ at the point B, $OB \perp \gamma$. Since $\alpha \perp \gamma$ and $\beta \perp \gamma$, the line of intersection of the planes α and β is also perpendicular to the plane γ. Let the planes α and β intersect along a straight line AC. Then AC is perpendicular to γ, and, consequently, OB is parallel to AC. The straight lines OB and AC define a certain plane σ, and, since the straight line OB lies in the plane σ, σ is perpendicular to γ. Let us take the plane γ for *horizontal projection plane* and denote it by H (retaining the notation used in descriptive geometry), and the plane σ for the *vertical projection plane* and denote it by V.

Let us also note the following: since the circles ω_1 and ω_2 are congruent, the point O is equidistant from the planes α and β, and, consequently, V is the bisecting plane of the dihedral angle made by the planes α and β, and the plane V intersects the sphere Σ along the circumference of the great circle. The pair of orthogonal projections of the sphere Σ and the trihedral angle $\alpha\beta\gamma$ is represented in Fig. 157.

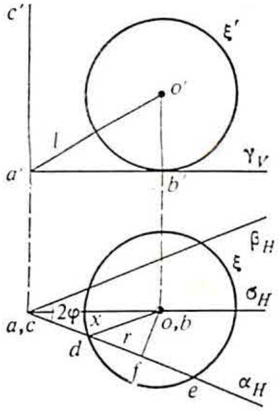


Fig. 157

Thus, in the plane V we see the section of the sphere Σ by the plane σ passing through the point O, i.e. the centre of the sphere, and the vertex of the given trihedral angle, and, therefore, $a'o' = l$. In the plane H we see a true-size representation of the dihedral angle $\alpha\beta$, therefore, $\angle\alpha_H\beta_H = 2\varphi$ and $\angle\alpha_H\sigma_H = \varphi$, the line segment de is the representation of the horizontal projection of the circle ω_1, that is, $de = 2r$, and, finally, the desired radius of the sphere Σ is equal to the radius of the circles ξ and ξ', i.e. the horizontal and vertical projections of the sphere Σ, respectively.

In the plane H we drop the perpendicular from the point of to α_H (the line segment of is the representation of the line segment OF which is perpendicular to α). Then $df = r$. For the sake of brevity, let us set the radius of the sphere Σ equal to x. Then the radii of the circles ξ and ξ' are also equal to x, that is, $od = x$ and $o'b' = x$.

Then we find from the right triangle $o'a'b'$ that $a'b' = \sqrt{l^2 - x^2}$. But it is not difficult to note that $ao = a'b'$. Thus, $ao = \sqrt{l^2 - x^2}$ too. Then, since $\angle aof = \varphi$, we find from the right triangle aof that $of = ao \sin\varphi = \sqrt{l^2 - x^2}\sin\varphi$, and, consequently, we get from the right triangle odf:

$$x^2 = r^2 + (l^2 - x^2)\sin^2\varphi. \tag{1}$$

Solving this equation with respect to x^2, we obtain: $x^2 = \frac{r^2 + l^2\sin^2\varphi}{1 + \sin^2\varphi}$. Rejecting at once the negative value of x, as it is a fortiori extraneous, we get: $x = \sqrt{\frac{r^2 + l^2\sin^2\varphi}{1 + \sin^2\varphi}}$.

Now, let us determine the constraints (bearing in mind the sense of the problem) to be satisfied by r, l, and φ for the found value of x (one of the solutions of the quadratic equation (1)) to be the length of the radius of the sphere Σ.

Firstly, since 2φ is the angle between the planes α and β, we have: $0° < 2\varphi < 90°$, that is, $0° < \varphi < 45°$.

Secondly,

$$\begin{cases} of < x < ao, \\ \qquad x > df, \end{cases} \quad \text{or} \quad \begin{cases} \sqrt{l^2-x^2}\sin\varphi < x < \sqrt{l^2-x^2}, \\ \qquad x > r. \end{cases}$$

Substituting the value of x into this system and solving the obtained system of inequalities, we get: $r < \frac{\sqrt{2}}{2} l \cos\varphi$.

$$\text{Thus, } OB = \sqrt{\frac{r^2+l^2\sin^2\varphi}{1+\sin^2\varphi}}, \text{ where } \begin{cases} r < \frac{\sqrt{2}}{2} l\cos\varphi, \\ 0° < \varphi < 45°. \end{cases}$$

Remark. In the expression found for the value of x, the parameters r, l, and φ may seemingly attain any real values (the radicand is positive), however, the investigation carried out in accordance with the sense of the problem showed that this is by far not so.

Example 4. The centre of the ball inscribed in a regular quadrangular pyramid coincides with the centre of the ball circumscribed about this pyramid. Find the dihedral angle at the edge of the base of the pyramid.

Solution. Let the figure $SABCD$ be the representation of the regular quadrangular pyramid (Fig. 158). This is a complete representation, and its parametric number $p=4$. We construct SP, i.e. the slant height of the lateral face SAB, and OP, i.e. the projection of the slant height SP on the plane ABC. We shall assume that PK is the representation of the bisector of the angle SPO, that is, we shall assume the point K to be the representation of the centre of the inscribed ball Ω. Thus, one more parameter is spent on the representation, and the latter becomes metrically determined. Since, by the hypothesis, the point K is also the centre of the circumscribed ball Σ, one should hold that AK and SK are the representations of two line segments congruent in the original, that is, $AK = SK$. There is no necessity to represent the balls Ω and Σ. Thus, OK is the radius of the inscribed ball, AK is the radius of the circumscribed ball, and it is required to find the dihedral angle at the edge AB of the pyramid, that is, the angle $SABO$.

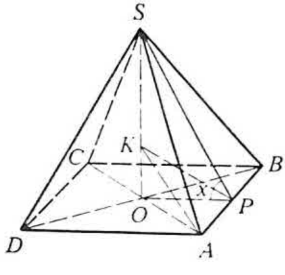


Fig. 158

Since SP is the altitude of the triangle SAB in which $SA=SB$, we have: $SP \perp AB$. But OP is the projection of the line segment SP on the plane ABC. Then OP is also perpendicular to AB. Thus, $\angle SPO$ is the plane angle of the sought-for dihedral angle $SABO$. Let us set, for brevity, that $\angle SPO = x$ and introduce an auxiliary parameter to facilitate all necessary computations, putting $AB=a$. We express SO in two different ways. We find from the right triangle SOP that $SO=\frac{a}{2}\tan x$. Further, we find from the right triangle POK that $OK=\frac{a}{2}\tan\frac{x}{2}$, and from the right triangle AOK that $AO=\frac{a\sqrt{2}}{2}$. Then $AK=\sqrt{AO^2+OK^2}=\frac{a}{2}\sqrt{2+\tan^2\frac{x}{2}}$. But $SK=AK$, that is, $SK=\frac{a}{2}\sqrt{2+\tan^2\frac{x}{2}}$. Since $SO=OK+SK$, we have: $SO=\frac{a}{2}\left(\tan\frac{x}{2}+\sqrt{2+\tan^2\frac{x}{2}}\right)$. Thus, $\frac{a}{2}\tan x=\frac{a}{2}\left(\tan\frac{x}{2}+\sqrt{2+\tan^2\frac{x}{2}}\right)$, whence $\tan x-\tan\frac{x}{2}=\sqrt{2+\tan^2\frac{x}{2}}$. Since $\frac{x}{2}$ is an acute angle, we have: $\frac{\sin\frac{x}{2}}{\cos x\cdot\cos\frac{x}{2}}=\frac{\sqrt{1+\cos^2\frac{x}{2}}}{\cos\frac{x}{2}}$. Further transformations yield: $\sin\frac{x}{2}=\cos x\sqrt{1+\cos^2\frac{x}{2}}$. Squaring both sides of this equation $\sin^2\frac{x}{2}=\cos^2 x\left(1+\cos^2\frac{x}{2}\right)$, passing to $\cos\frac{x}{2}$, and denoting, for brevity, that $\cos^2\frac{x}{2}=y$, we get the equation: $1-y=(2y-1)^2(1+y)$, which, after a number of simplifications, is transformed to the equation: $4y^3-2y=0$.

Thus, we find that $y_1=\frac{\sqrt{2}}{2}$, $y_2=-\frac{\sqrt{2}}{2}$, and $y_3=0$. Rejecting the extraneous values y_2 and y_3, we get: $\cos^2\frac{x}{2}=\frac{\sqrt{2}}{2}$, i.e. $\frac{1+\cos x}{2}=\frac{\sqrt{2}}{2}$, whence $\cos x=\sqrt{2}-1$, and, consequently, $x=\arccos(\sqrt{2}-1)$. Thus, $\angle SABO=\arccos(\sqrt{2}-1)$.

PROBLEMS TO BE SOLVED WITHOUT ASSISTANCE

853. The side of a rhombus is equal to a. A sphere whose radius is R touches all the sides of the rhombus. The distance from the centre of the sphere to the plane of the rhombus is equal to α. Find the area of the rhombus.

854. Given on the surface of a sphere of radius R are two equal circles whose common chord is equal to a. Find the radii of these circles if their planes are mutually perpendicular.

855. A ball of radius R is inscribed in a cone whose generatrix is seen from the centre at an angle equal to α. Find the volume of the cone.

856. Inscribed in a hemisphere of radius R is a frustum of a cone so that its larger base coincides with the base of the hemisphere, and the generatrix is inclined at an angle equal to α to the plane of the base. Find the surface area of the cone.

857. In a right circular cone, the area of the base is equal to S_1, and the area of the lateral surface to S_2. Find the radius of the ball inscribed in the cone.

858. A ball is inscribed in a cone so that the radius of the circle of its tangency with the cone is equal to R. A straight line passing through the centre of the ball and a point lying on the circle of tangency makes an angle equal to α with the plane of the base. Find the volume of the cone.

859. Find the vertex angle in the axial section of a cone if it is known that it is possible to draw three pairwise perpendicular elements on its surface.

860. Two equal cones with an angle equal to α at the vertex of the axial section are arranged so that the axis of each of them is an element of the other. Find the angle between two elements along which the cones intersect.

861. A plane parallel to the base of the cone and passing through the centre of the ball inscribed in this cone divides the cone into two parts whose volumes are equal to each other. Find the angle between the generatrix of the cone and the plane of its base.

862. A hemisphere is inscribed in an equilateral cone so that its great circle is found in the plane of the base of the cone. Find the ratio in which the circle of tangency divides the lateral surface of the hemisphere and the lateral surface of the cone.

863. Two balls are placed into a cone so that they touch each other and the surface of the cone. The ratio of the radii of these balls is equal to $m:n$ $(m > n)$. Find the angle at the vertex of the axial section of the cone.

864. Prove that the ratio of the volume of a cone to the volume of the ball inscribed in it is equal to the ratio of the total surface area of the cone to the surface area of the ball.

865. Inscribed in a frustum of a cone is a ball whose volume is $\frac{6}{13}$ of the volume of the cone. Find the angle between the generatrix of the cone and the plane of its lower base.

866. Inscribed in a hemisphere is a cone whose vertex coincides with the centre of a circle, which is the base of the hemisphere. The plane of the base of the cone is parallel to the plane of the base of the hemisphere. A straight line joining the centre of the base of the cone to an arbitrary point on the great circle of the hemisphere makes an angle equal to α with the plane of the base of the cone. Find the ratio of the volume of the hemisphere to the volume of the cone.

867. A ball is inscribed in a cone. The line of their tangency divides the surface area of the ball in the ratio $m:n$. Find the angle between the generatrix of the cone and its axis.

868. Inscribed in a cube is a pyramid, one of whose vertices is the centroid of a face of the cube, the four others being the vertices of the opposite face of the cube. A ball is inscribed in the pyramid. In what ratio does the plane passing through the centre of the ball parallel to the base of the pyramid divide the volume of the cube?

869. A ball touches the lateral surface of a cone along the circle of the base. The surface area of the ball is divided thereby into two parts, one of which is n times greater than the other. Find the angle between the generatrix of the cone and the plane of its base.

870. Circumscribed about a cone is a triangular pyramid. The lateral surface of the cone is divided by the lines of tangency into parts whose areas are to one another as 5:6:7. In what ratio is the lateral surface area of the pyramid divided by the same lines?

871. Inscribed in a cone is a cylinder, the total surface area of which is equal to the lateral surface area of the cone. The angle between the elements of the cone in its axial section is equal to 90°. Prove that the distance from the vertex of the cone to the upper base of the cylinder is equal to half the length of the generatrix of the cone.

872. Two cones have a common base. In a common axial section, the element of one of them is perpendicular to the opposite element of the other. The volume

of one of the cones is half the volume of the other. Find the angle between the generatrix of the larger cone and the plane of the bases of the cones.

873. Drawn from a point taken on the surface of a ball are three equal chords, the angle between each pair of which is equal to α. Find the lengths of the chords if the radius of the ball is equal to R.

874. The base of a pyramid is an isosceles triangle, each of the lateral sides of which is equal to a, the angle between them being equal to α. Two lateral faces are perpendicular to the plane of the base, and the third face forms an angle equal to β with it. Find the radius of the ball inscribed in the pyramid.

875. A sphere is passed through the midpoints of the lateral edges of a cube and touches one of its bases. What part of the volume of the cube lies inside the sphere?

876. A ball is inscribed in a cube with edge a so that its surface touches all the edges of the cube. Find the volume of the part of the ball enclosed inside the cube.

877. The edge of a cube is equal to a. Find the radius of two equal balls, which can be placed in the cube so that they cannot move inside the cube when the latter is displaced.

878. A ball is inscribed in a cube with edge a. Inscribed in one of the trihedral angles at the vertex of the cube is another ball, which touches the first ball. Find the radius of the second ball.

879. A ball is passed through the vertices A, B, and D of the cube $ABCDA_1B_1C_1D_1$ and the midpoint of the edge A_1B_1. Find the radius of the ball if the edge of the cube is equal to a.

880. A ball touches three faces of a cube containing one vertex and three edges of this cube containing the opposite vertex. Find the edge of the cube if the radius of the ball is equal to R.

881. A ball touches three faces of a cube containing one vertex and is passed through the vertex of the cube opposite to the first one. Find the radius of the ball if the edge of the cube is equal to a.

882. A ball touches three edges of a cube containing one vertex and is passed through the vertex of the cube opposite to the first one. Find the radius of the ball if the edge of the cube is equal to a.

883. A ball is passed through the midpoints of three edges of a cube containing one vertex and through the vertex of the cube opposite to the first one. Find the edge of the cube if the radius of the ball is equal to R.

884. A ball touches four edges of a cube belonging to one of its faces and to the opposite face. Find the ratio of the volume of the part of the ball lying outside the cube to the volume of the ball.

885. A ball is passed through the vertices of the lower base of a cube and touches the edges of its upper base. Find the ratio of the edge of the cube to the radius of the ball.

886. A ball touches all the lateral edges of a regular quadrangular prism and its bases. Find the ratio of the surface area of the ball lying outside the prism to the total surface area of the prism.

887. A ball is inscribed in a right prism whose base is a right triangle in which the perpendicular dropped from the vertex of the right angle to the hypotenuse is equal to h and makes an angle equal to α with one of the legs. Find the volume of the prism.

888. In a regular prism $ABCA_1B_1C_1$, drawn through the side AB of the base is a plane also passing through the vertex C_1 of the other base. A ball is inscribed in the pyramid $C_1ABB_1A_1$ (C_1 the vertex). Find the angle between the plane ABC_1 and the plane of the base of the prism.

889. Circumscribed about a ball is a right parallelepiped whose volume is m times the volume of the ball. Find the base angles of the parallelepiped.

890. The edge of a regular tetrahedron is equal to a. Find the radius of the ball touching the lateral faces of the tetrahedron at the vertices of the base.

891. The edge of a regular tetrahedron is equal to a. Find the radius of the ball touching the lateral faces of the tetrahedron at points lying on the sides of the base.

892. Prove that if the vertices of the lower base of a right triangular prism lie on the surface of a ball, and the sides of the upper base touch this ball, then the prism is regular.

893. A ball touches all the lateral faces of a triangular pyramid at the centres of the circles circumscribed about them. Each plane angle at the vertex of the pyramid is equal to 2α, the sum of the lateral edges being equal to $3a$. Find the radius of the ball.

894. The altitude of a triangular pyramid is equal to H, and the sum of all nine plane angles at the vertices of the base is equal to α. Find the radius of the ball, which touches all the lateral faces at the points of intersection of their medians.

895. The lateral surface of a triangular pyramid is equal to S, and the side of its base is equal to a. A ball touches three edges of the base at their midpoints and intersects the lateral edges at their midpoints. Find the radius of the ball.

896. Each of the plane angles at the vertex S of a pyramid is equal to 90°. Prove that the vertex S, i.e. the centre of the ball circumscribed about the pyramid, and the point of intersection of the medians of the base ABC lie in one straight line.

897. A ball of radius R is inscribed in a pyramid, each face of the pyramid being inclined at an angle equal to α. Find the volume of the pyramid if its base is a rhombus whose acute angle is equal to β.

898. In a regular pyramid $SABCD$ with vertex S, each side of the base is equal to a, and the lateral edge is equal to b. The first sphere centred at O_1 touches the planes SAD and SBC at points A and B, respectively, and the second sphere centred at O_2 touches the planes SAB and SCD at points B and C, respectively. Find the volume of the pyramid ABO_1O_2.

899. Find the radius of the ball inscribed in a regular quadrangular pyramid if the volume of the pyramid is equal to V, the angle between its two opposite faces being equal to α.

900. The base of a pyramid is an isosceles triangle, each of the congruent angles of which is equal to α, the common side of these angles being equal to a. Each of the lateral faces of the pyramid is inclined at an angle equal to β to the plane of the base. Find the radius of the ball inscribed in the pyramid.

901. Inscribed in a ball of radius R is a pyramid whose base is a square. One of the lateral edges is perpendicular to the plane of the base, and the largest lateral edge forms an angle equal to α with it. Find the lateral surface area of the pyramid.

902. In a regular quadrangular pyramid, the plane angle at the vertex is equal to α, and the altitude of the pyramid is equal to H and serves as the diameter of a ball. Find the length of the line of intersection of the surfaces of the pyramid and the ball.

903. Placed in a regular quadrangular pyramid are two balls touching each other and all the lateral faces of the pyramid. The lower ball also touches the base of the pyramid. The ratio of the radius of the larger ball to the radius of the smaller ball is equal to n. Find the dihedral angles of the pyramid.

904. A ball is inscribed in a regular triangular pyramid, the plane angle at the vertex of which is equal to α. Into what parts is the surface area of the ball divided by the plane passing through the points of tangency of the ball and the lateral faces of the pyramid?

905. A lateral edge of a regular quadrangular pyramid is equal to b, and the angle made by the lateral edge and the plane of the base is equal to α. Inscribed in this pyramid is an equilateral cylinder so that one of its elements is situated on the diagonal of the base of the pyramid, and the circle of the base touches two adjacent lateral faces of the pyramid. Find the radius of the base of the cylinder.

906. Inscribed in a cylinder whose altitude is equal to H is a triangular pyramid. Two faces of the pyramid are perpendicular to the plane of its base, and two lateral edges make angles, each of which is equal to α with the plane of the base. The angle between these edges is equal to β. Find the lateral surface area of the pyramid.

907. Each edge of a regular tetrahedron is equal to a. A cylindrical surface passes through one of its edges and all of its vertices. Find the radius of the base of the cylinder.

908. The bases of a spherical segment and a cylinder coincide. The volume of the solid enclosed between their lateral surfaces is equal to 36π cm^3. Find the altitude of the cylinder, which is equal to the altitude of the spherical segment.

909. Inscribed in a cone with radius of the base equal to R is a triangular prism with equal edges so that its base lies in the plane of the base of the cone. Find the volume of the prism if the angle between the generatrix of the cone and the plane of its base is equal to α.

910. Inscribed in a cube whose edge is equal to a is a right circular cone with the angle between the elements in the axial section equal to α. Find the length of the generatrix and the radius of the base of the cone if its altitude lies on the diagonal of the cube.

911. A cube $ABCDA_1B_1C_1D_1$ is inscribed in a cone, the radius of the base of which is equal to R and the altitude is equal to $R\sqrt{2}$. The base $ABCD$ of the cube lies in the base of the cone, and the points A_1, B_1, C_1, and D_1 are found on its lateral surface. Find the area of the triangle A_1C_1M, where M is the point of intersection of the straight line BD and the circle of the base.

912. Two cones have concentric bases and a common altitude, which is equal to H. The difference between the angles formed by the elements and the axis is equal to β, and the angle between the generatrix of the inner cone and the plane of the base is equal to α. Find the volume of the part of space contained between the surfaces of the cones.

913. The sides of an isosceles trapezoid touch a cylinder whose axis is perpendicular to the parallel sides of the trapezoid. Find the angle between the plane of the trapezoid and the axis of the cylinder if the bases of the trapezoid are equal to a and b $(a > b)$, its altitude being equal to h.

914. Constructed on a rectangular sheet of paper, one side of which is equal to a, are circles whose radii are $\frac{a}{12}$ and $\frac{a}{4}$. The distance between the centres of the circles is equal to $\frac{2a}{3}$, and the line of their centres is parallel to the base of the rectangle. A common internal tangent is drawn to the circles. Find the distance between the points of tangency after the sheet has been rolled up to form a circular cylindrical surface whose axis is perpendicular to the line of centres of the circles.

915. Given a cube and a regular quadrangular pyramid whose lateral edge is equal to b. The vertices of one of the faces of the cube serve as the midpoints of the edges of the bases of the pyramid, and each edge of the opposite face of the cube intersects one of the lateral edges of the pyramid. Find the volume of the part of the cube situated outside the pyramid.

916. In a cube whose edge is equal to a a diagonal is drawn. The edges of the cube converging at one end point of the diagonal are bisected. The division points thus obtained and the other end point of the diagonal are taken for the vertices of the pyramid. Find the volume of this pyramid.

917. The edges of a triangular pyramid emanating from the vertex A are pairwise perpendicular and equal to a, b, and c. Find the volume of the cube inscribed in the pyramid so that one of its vertices coincides with the vertex A.

918. A lateral edge of a regular triangular pyramid is equal to b and forms an angle equal to α with the plane of the base. Inscribed in the pyramid is an equilateral cylinder so that its lower base lies in the plane of the base of the pyramid. Find the altitude of the cylinder.

919. On a sheet of paper, which is a square $PQML$, a hole is cut, having the shape of an equilateral triangle ABC, so that $AB \parallel PL$ and $AB:PL = 1:2$. Then the square is rolled up to form a circular cylindrical surface whose axis is perpendicular to the line segment AB. Find the ratio of the area of the square to the area of the triangle ABC whose vertices lie on the cylindrical surface.

SEC. 16. GREATEST AND LEAST VALUES

When solving stereometric problems on finding greatest and least values, we shall stick to the same scheme, which was followed when solving similar planimetric problems (see Sec. 7, Chapter 1).

Example 1. A cube is cut by a plane passing through one of its diagonals. Prove that the section forming an angle of $\arccos \frac{\sqrt{6}}{3}$ with the plane of the base has the least value.

Solution. The quantity to be optimized is the area S of the section. Let the edge of the cube be equal to a (a known quantity), and B_1KDL be a certain section (Fig. 159). We introduce an independent variable: $C_1K = x$. According to the sense of the problem, it is clear that $0 \leqslant x \leqslant a$ are real limits within which x varies.

We express the area S in terms of x and a. Note, first of all, that the obtained section is a parallelogram, since the lines of intersection of two parallel planes with a third plane are parallel to each other. The area of the parallelogram can be found by the formula $B_1K \cdot KD \cdot \sin\alpha$, where $\alpha = \angle B_1KD$. We find from the triangle B_1C_1K that $B_1K = \sqrt{a^2+x^2}$, and we find from the triangle DKC that $DK = \sqrt{a^2+(a-x)^2}$. By the law of cosines, we get from the triangle B_1KD: $B_1D^2 = B_1K^2 + KD^2 - 2B_1K \cdot KD \cdot \cos\alpha$, that is, $(a\sqrt{3})^2 = (a^2+x^2) + (a^2+(a-x)^2) - 2\sqrt{a^2+x^2}\sqrt{a^2+(a-x)^2}\cos\alpha$, whence, after a number of transformations, we get: $\cos\alpha = \frac{x^2-ax}{\sqrt{a^2+x^2}\sqrt{a^2+(a-x)^2}}$. Then $\sin\alpha =$

$$\sqrt{1-\cos^2\alpha} = \sqrt{1-\frac{(x^2-ax)^2}{(a^2+x^2)(a^2+(a-x)^2)}} = \frac{a\sqrt{2a^2+2x^2-2ax}}{\sqrt{a^2+x^2}\sqrt{a^2+(a-x)^2}}.$$

Finally, we get: $S = B_1K \cdot KD \cdot \sin\alpha = \sqrt{a^2+x^2}\sqrt{a^2+(a-x)^2} \times$

$$\frac{a\sqrt{2a^2+2x^2-2ax}}{\sqrt{a^2+x^2}\sqrt{a^2+(a-x)^2}} = a\sqrt{2a^2+2x^2-2ax}.$$

We have to find the least value of the function $S(x) = a\sqrt{2a^2+2x^2-2ax}$ on $[0, a]$. The derivative $S' = a\frac{2x-a}{\sqrt{2a^2+2x^2-2ax}}$.

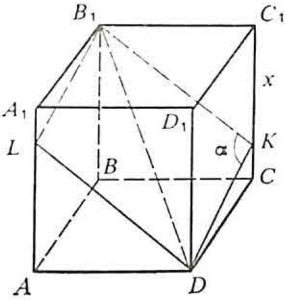


Fig. 159

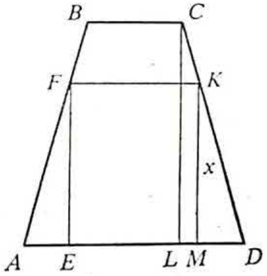


Fig. 160

In the present case, there are no points at which S' does not exist, since the denominator of the derivative vanishes nowhere (if by the hypothesis, $a \geqslant x$, then $a^2 \geqslant ax$ and $2a^2 + 2x^2 - 2ax > 0$). The derivative $S' = 0$ if $2x - a = 0$, that is, for $x = \frac{a}{2}$.

To find the least value of the function, it remains to compute the values of the function $S(x)$ at the end points of the interval $[0, a]$ and compare them with the value of the function at the point $x = \frac{a}{2}$ We have: $S(0) = a^2\sqrt{2}$, $S(a) = a^2\sqrt{2}$, and $S\left(\frac{a}{2}\right) = \frac{a^2\sqrt{6}}{2}$. The least value of the function $S(x)$ equals $\frac{a^2\sqrt{6}}{2}$. It is reached for $x = \frac{a}{2}$.

In the problem it is required to prove that the angle formed by the least (by area) section and the plane of the base is equal to $\arccos\frac{\sqrt{6}}{3}$. To prove this fact, we use the formula $S_{\text{base}} = S_{\text{sec}} \cos\varphi$, where φ is the angle between the planes of the section and of the base. We have: $a^2 = \frac{a^2\sqrt{6}}{2}\cos\varphi$, whence we find that $\cos\varphi = \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}$, that is, $\varphi = \arccos\frac{\sqrt{6}}{3}$.

Example 2. A 20-dm long log has the shape of a frustum of a cone with diameters of the bases of 2 dm and 1 dm. It is required to cut out a squared beam whose axis would coincide with the axis of the log and whose volume would be the greatest possible.

Solution. The quantity to be optimized here is the volume V of the beam, that is, the volume of a rectangular parallelepiped with a square base.

In an axial section of the cone, which is simultaneously a diagonal section of the rectangular parallelepiped, we get (Fig. 160) an equilateral trapezoid (an axial section of the frustum of the cone) in which a rectangle (the diagonal section of the rectangular parallelepiped) is inscribed. We denote by x the altitude of the parallelepiped, that is, the altitude of the rectangle in the axial section: $KM = x$, x varying within the limits of $0 < x \leqslant 20$.

Let us find the volume V of the rectangular parallelepiped. The line segment FK represents the diagonal of the base of the parallelepiped. We find FK. We have: $FK = EM = AD - 2MD = 2 - 2MD$. We draw CL perpendicular to AD. Then $LD = AD - AL = 1 - 0.5 = 0.5$ dm. Since the triangles KMD and CLD are similar, we have: $\frac{KM}{CL} = \frac{MD}{LD}$, i.e. $\frac{x}{20} = \frac{MD}{0.5}$, whence we find that $MD = \frac{x}{40}$, and, hence, $FK = 2 - 2MD = 2 - \frac{x}{20}$.

The area of the square serving as the base of the rectangular parallelepiped can be found by the formula $\frac{1}{2} d^2$, where d is the diagonal of the base, that is, $d = FK$. Hence, $S_{\text{base}} = \frac{1}{2}\left(2 - \frac{x}{20}\right)^2$. Since the altitude of the parallelepiped is equal to x, we get the following result for the volume: $V = \frac{1}{2}\left(2 - \frac{x}{20}\right)^2 x$.

For the function $V = \frac{1}{2}\left(2 - \frac{x}{20}\right)^2 x$ we have to find the greatest value on the interval $(0, 20]$. We have: the derivative $V' = 2\left(2 - \frac{x}{20}\right)\left(-\frac{1}{20}\right) x + \left(2 - \frac{x}{20}\right)^2 = \left(2 - \frac{x}{20}\right)\left(2 - \frac{3x}{20}\right)$. The derivative $V' = 0$ for $x = 40$ or for $x = \frac{40}{3}$. The value $x = 40$ does not belong to the interval under consideration.

x	0	$\frac{40}{3}$	20
V	0	$\frac{320}{27}$	10

The greatest value of the function equals $\frac{320}{27}$.

We now interpret the obtained result for this problem. To cut the beam of the greatest volume, we have to remove the upper (the thinner) part of the log so as to make the length of the log equal to $13\frac{1}{3}$ dm and then from the log thus obtained to cut out a square beam $\left(\text{its cross-section is determined by the square inscribed in the upper base of the log, } 13\frac{1}{3}\text{-dm long}\right)$.

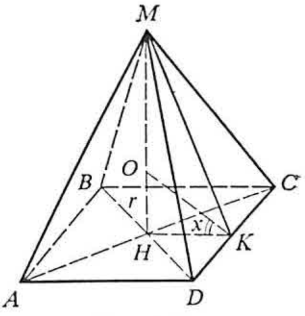


Fig. 161

Example 3. A regular quadrangular pyramid is circumscribed about a ball of radius r. Find the least value of its lateral surface area.

Solution. Here, the quantity to be optimized is the lateral surface area S. We introduce an independent variable. Let us recall that the centre O of the inscribed ball lies on the altitude of the regular pyramid, namely, at the point of intersection of the altitude and the bisector OK of the angle between the slant height MK of the lateral face and the projection HK of the slant height on the plane of the base (Fig. 161), $OH = r$ being the radius of the inscribed ball. We set $\angle OKH = x$, x varying within the limits of $0 < x < \frac{\pi}{4}$.

We express S in terms of r and x. We find from the triangle OKH that $HK = r \cot x$, we find from the triangle HKD that $KD = HK = r \cot x$, and we find from the triangle MHK that $MK = \frac{HK}{\cos 2x} = \frac{r \cot x}{\cos 2x}$. Then $S = 4S_{\triangle MCD} = 4KD \cdot MK = 4r \cot x \, \frac{r \cot x}{\cos 2x} = 4r^2 \frac{\cot^2 x}{\cos 2x}$.

We now find the least value of the function $S = 4r^2 \frac{\cot^2 x}{\cos 2x}$ on the interval $\left(0, \frac{\pi}{4}\right)$. We have:

$$S' = 4r^2 \frac{2 \cot x \left(-\frac{1}{\sin^2 x}\right) \cos 2x + 2 \sin 2x \cot^2 x}{\cos^2 2x}$$

$$= \frac{8r^2}{\cos^2 2x} \cot x \left(\cot x \sin 2x - \frac{\cos 2x}{\sin^2 x}\right).$$

The derivative S' does not exist if $\cos 2x = 0$ or $\sin x = 0$ which is not fulfilled on the interval $\left(0, \frac{\pi}{4}\right)$.

The derivative $S' = 0$ if $\cot x = 0$ which is not fulfilled on the interval $\left(0, \frac{\pi}{4}\right)$ or if $\cot x \sin 2x - \frac{\cos 2x}{\sin^2 x} = 0$. Let us solve this trigonometric equation. We have in succession:

$$\frac{\cos x \sin 2x}{\sin x} - \frac{\cos 2x}{\sin^2 x} = 0,$$

$$\sin x \cos x \sin 2x - \cos 2x = 0, \quad \sin^2 2x - 2 \cos 2x = 0,$$

$$1 - \cos^2 2x - 2 \cos 2x = 0, \quad \cos^2 2x + 2 \cos 2x - 1 = 0,$$

$$(\cos 2x)_{1,2} = -1 \pm \sqrt{2}.$$

The only suitable value is $\cos 2x=\sqrt{2}-1$, whence it follows that $x=\frac{1}{2}\arccos(\sqrt{2}-1)$.

To make sure that for the found value of x the function $S(x)$ reaches the least value, we compute the one-sided limits of the function at the end points of the interval $\left(0, \frac{\pi}{4}\right)$:

$$\lim_{x\to 0} S(x) = 4r^2 \lim_{x\to 0} \frac{\cot^2 x}{\cos 2x} = +\infty,$$

$$\lim_{x\to \frac{\pi}{4}+0} S(x) = 4r^2 \lim_{x\to \frac{\pi}{4}+0} \frac{\cot^2 x}{\cos 2x} = +\infty$$

Hence, at the obtained point, the function $S(x)$ actually has the least value. Let us compute this value. We get: $S=4r^2\frac{\cot^2 x}{\cos 2x}$ and $\cos 2x=\sqrt{2}-1$. Since $1+\cot^2 x=\frac{1}{\sin^2 x}=\frac{2}{1-\cos 2x}$, we have: $\cot^2 x=\frac{2}{1-\cos 2x}-1=\frac{2}{1-(\sqrt{2}-1)}-1=\sqrt{2}+1$. Hence, $S_{\text{least}}=4r^2\frac{\sqrt{2}+1}{\sqrt{2}-1}=4r^2(\sqrt{2}+1)^2$.

Returning to the original geometrical problem, we conclude that the least possible value of the lateral surface area of the regular quadrangular pyramid circumscribed about the ball of radius r is equal to $4r^2(\sqrt{2}+1)^2$.

Example 4. An n-gonal pyramid is inscribed in a ball. For what dihedral angle between a lateral face and the plane of the base of the pyramid will the volume of the pyramid be the greatest?

Solution. The quantity to be optimized is the volume V of the pyramid. Let us introduce an independent variable. We represent in Fig. 162 the nth part of the pyramid: MAB is a lateral face and MO is the altitude. We denote the radius of the ball by R (an auxiliary parameter) and set $MB = x$, x varying within the limits of $0 < x < 2R$.

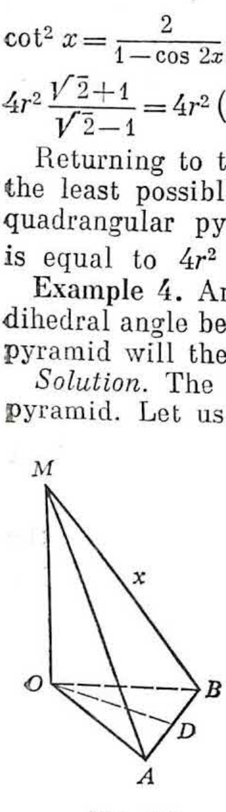


Fig. 162

We express V in terms of x and R and take advantage of the known formula for computing the circumscribed ball in the case of a regular pyramid: $R=\frac{b^2}{2H}$, where b is a lateral edge, and H is the altitude of the pyramid. Then $R=\frac{x^2}{2H}$, whence $H=\frac{x^2}{2R}$. We find from the triangle MBO that $OB=\sqrt{MB^2-MO^2}=$

$\sqrt{x^2-\frac{x^4}{4R^2}}$. Since AB is the side of a regular n-gon, we have: $\angle AOB=\frac{2\pi}{n}$, and, therefore, $S_{\Delta AOB}=\frac{1}{2}AO\cdot OB\cdot\sin\frac{2\pi}{n}=\frac{1}{2}\left(x^2-\frac{x^4}{4R^2}\right)\sin\frac{2\pi}{n}$. Hence, $V=\frac{1}{3}S_{\text{base}}H=\frac{1}{3}n\frac{1}{2}\left(x^2-\frac{x^4}{4R^2}\right)\times$

$$\sin\frac{2\pi}{n}\cdot\frac{x^2}{2R}=\frac{n\sin\frac{2\pi}{n}}{12R}\left(x^4-\frac{x^6}{4R^2}\right).$$

Let us find the greatest value of the function $V=k\left(x^4-\frac{x^6}{4R^2}\right)$ on the interval $(0,\ 2R)$ $\left(\text{for brevity, we set: } k=\frac{n\sin\frac{2\pi}{n}}{12R}\right)$. We have: $V'=k\left(4x^3-\frac{6x^5}{4R^2}\right)$. We find from the equation $V'=0$ that $x_1=0,\ x_2=R\sqrt{\frac{8}{3}}$, and $x_3=-R\sqrt{\frac{8}{3}}$. Of these three values only $x=R\sqrt{\frac{8}{3}}$ belongs to the interval $(0,\ 2R)$. We have: $\lim\limits_{x\to 0}V(x)=\lim\limits_{x\to 2R}V(x)=0$. Hence, for $x=R\sqrt{\frac{8}{3}}$ the function $V(x)$ reaches the greatest value equal to $\frac{16R^3n\sin\frac{2\pi}{n}}{27}$.

In the problem it is required to find the angle MDO for the pyramid having the greatest volume, where MD is the slant height of the lateral face. Setting $\angle MDO=\varphi$, we have: $MO=\frac{x^2}{2R}=\frac{4R}{3}$ $\left(x\text{ is substituted by its value } R\sqrt{\frac{8}{3}}\text{ found above}\right)$. We find from the triangle BDO that $OD=OB\cos\frac{\pi}{n}=\sqrt{x^2-\frac{x^4}{4R^2}}\cos\frac{\pi}{n}=\frac{2R\sqrt{2}}{3}\cos\frac{\pi}{n}$. Then $\tan\varphi=\frac{SO}{DO}=\frac{\sqrt{2}}{\cos\frac{\pi}{n}}$.

Hence, $\varphi=\arctan\left(\frac{\sqrt{2}}{\cos\frac{\pi}{n}}\right)$.

PROBLEMS TO BE SOLVED WITHOUT ASSISTANCE

920. Prove that of all regular quadrangular pyramids in which the sum of the altitude and the side of the base is constant, the greatest volume is possessed by the pyramid, whose lateral face makes an angle equal to 45° with the plane of the base.

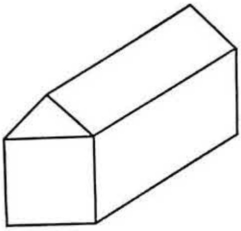


Fig. 163

921. The area of the base of a rectangular parallelepiped is equal to 1 cm^2, and the length of the diagonal is equal to 2 cm. Find: (a) the greatest volume; (b) the greatest lateral surface area.

922. The sum of the squares of the lengths of all the edges of a regular triangular pyramid is equal to P. Find the greatest value of the lateral surface area of the pyramid.

923. Tha base of a pyramid $MABCD$ is a square, MB being the altitude of the pyramid. Find the least value of the length of the edge MD if the volume of the pyramid is equal to 9 cm^3.

924. Among regular n-gonal pyramids with a constant length of the lateral edge find the pyramid having the greatest volume (compute the angle between a lateral face and the plane of the base of the pyramid).

925. Among regular n-gonal pyramids with a constant area of the lateral face find the pyramid having the greatest volume (compute the angle between a lateral face and the plane of the base of the pyramid).

926. A little house is made of two right prisms with a square and a triangular base, the triangular base being a right isosceles triangle (Fig. 163). For what length of the side of the square will the volume of the house be the greatest if it is known that the perimeter of the base of the house is equal to 24 m?

927. Drawn in a triangular pyramid are sections parallel to two nonintersecting edges. Prove that the section passing through the midline of the base of the pyramid has the greatest area.

928. The base of a pyramid $MABC$ is a right isosceles triangle ABC ($AB = BC$). The faces MBC and MAB are perpendicular to the plane of the base, and $MC = 2\sqrt{2}$ cm. For what altitude of the pyramid will the area of the section passing through the points B and M and bisecting AC be the greatest?

929. Drawn through the diagonal of the base of a regular quadrangular prism is a section having at least one common point with another base. Find the greatest and the least area of such a section if the edges of the prism emanating from one vertex are equal to $3\sqrt{2}$ cm, $3\sqrt{2}$ cm, and 2 cm.

930. Prove that of all rectangular parallelepipeds with a square base inscribed in a hemisphere, a cube has the greatest volume.

931. Prove that of all pyramids whose base is an isosceles triangle and which is inscribed in a cone of a given volume, a regular pyramid has the greatest volume.

932. Find the greatest area of a section of a cone by a plane passing through its vertex if the radius of the base of the cone is equal to R, and the altitude is equal to H.

933. A cylinder is combined with a hemisphere. The volume of the solid thus obtained is equal to V. For what radius of the hemisphere will the total surface area of the solid be the least?

934. The perimeter of an isosceles triangle is $2p$. What must be its sides for the solid of revolution to have the greatest volume if this solid is generated by revolving the triangle about (a) the base; (b) the altitude drawn to the base?

935. A sector is cut from a given circle and is rolled up to form a conical funnel. What central angle of the sector must be chosen for the volume of the funnel to be the greatest?

936. In a given cone inscribe a cylinder of the greatest volume.

937. Inscribed in a ball of radius R is a cylinder. What is the altitude of the cylinder having: (a) the greatest volume; (b) the greatest lateral surface area?

938. For what altitude of a cone inscribed in a given ball of radius R: (a) will the volume of the cone be the greatest; (b) will the lateral surface area of the cone be the greatest?

939. Circumscribed about a ball of radius R is a cone. What is the altitude of the cone having: (a) the least volume; (b) the least lateral surface area?

940. Find the altitude of the cone of the least volume circumscribed about a hemisphere of radius R.

941. A cone of the greatest volume is inscribed in a given ball, and another ball is inscribed in the cone. Find the ratio of the volumes of the balls.

942. Inscribed in a ball is a cone of the greatest volume, and inscribed in the cone is a cylinder of the greatest volume. Find the ratio of the altitude of the cylinder to the radius of the ball.

943. Inscribed in a cone with a constant generatrix is a regular hexagonal prism, all the edges of which are equal. For what value of the angle between the generatrix of the cone and the plane of the base will the lateral surface area of the prism be the greatest?

944. In a regular quadrangular pyramid, a cylinder is inscribed so that the circle of its upper base touches all the lateral faces of the pyramid, and the lower base lies in the plane of the base of the pyramid. What part of the altitude of the pyramid must the altitude of the cylinder be equal to for the volume of the cylinder to be the greatest?

945. Inscribed in a hemisphere of radius R is a regular triangular prism so that one of its bases lies in the plane of the great circle of the hemisphere, and the vertices of the other base belong to the surface of the hemisphere. For what altitude of the prism will the sum of the lengths of all of its edges be the greatest?

946. Find the greatest volume of a regular hexagonal pyramid inscribed in a ball of radius R.

947. Circumscribed about a ball is a regular n-gonal pyramid with the least lateral surface area. Find the angle of inclination of its lateral edges to the plane of the base.

948. Inscribed in a ball is a regular quadrangular pyramid, and inscribed in the pyramid is a regular quadrangular prism so that one of the bases of the prism lies in the plane of the base of the pyramid, and the vertices of the other base of the prism belong to the lateral edges of the pyramid. The side of the base and the altitude of the prism are equal to $2a$ and a, respectively. Find the least value of the radius of the ball. For what altitude of the pyramid is this least value obtained?

949. The base of a pyramid $MABC$ is a right triangle ABC in which $\angle C = 90°$, $\angle A = 60°$, and $AC = 6$ cm. The edge MA is perpendicular to the plane of the base, MA being equal to 3 cm. Inscribed in the pyramid $MABC$ is a pyramid with vertex A whose base is the section of the given pyramid by the plane parallel to the edges MA and BC. What is the greatest volume of the inscribed pyramid?

950. The altitude of a regular quadrangular pyramid is twice the diagonal of its base, the volume of the pyramid being equal to V. Under consideration are regular quadrangular prisms inscribed in the pyramid so that their lateral edges are parallel to the diagonal of the base of the pyramid, one of their lateral faces belongs to this base, and the vertices of the opposite lateral face lie on the lateral surface of the pyramid. Find the greatest volume of the prism.

951. The volume of a regular quadrangular pyramid is equal to V, the angle between each lateral edge and the plane of the base being equal to 30°. A regular triangular prism is inscribed in the pyramid so that one of its lateral edges lies on the diagonal of the base of the pyramid, one of its lateral faces is parallel to the base of the pyramid, the vertices of this face lying on the lateral faces of the pyramid. Find the greatest volume of the prism.

ANSWERS AND HINTS

Chapter 1

2. $m, m\sqrt{3}$, and $2m$. 3. 56 cm and 42 cm. 4. $9\sqrt{5}$ cm and $8\sqrt{10}$ cm. 5. $\frac{ab\sqrt{2}}{a+b}$. 6. $\frac{mn(m+n)}{m^2+n^2}$. 7. 10 cm. 8. 15 cm, 20 cm, and 25 cm. 9. $\frac{m(m-n)\sqrt{2}}{\sqrt{m^2+n^2}}$ and $\frac{m(m+n)\sqrt{2}}{\sqrt{m^2+n^2}}$. 10. $\arctan\left(\frac{1}{2}\tan\alpha\right)-\frac{\alpha}{2}$. 11. $\frac{\pi}{4} \pm \arccos\frac{1+2\sqrt{2}}{4}$. Denote $\angle A=2x$ and $AB=c$. Express the legs and then the bisectors of the angles of the triangle in terms of c and x. 12. Set $\angle ACK=\angle ACB=\alpha$ (CK the altitude, CM a median, and $AC<BC$). $CK=h$, and $\angle ACB=x$. Express the line segments AK, BK, and MK in terms of h, α, and x. Take advantage of the fact that $2MK=BK-AK$. See also Example 10 from Sec. 3. 15. 6 cm. 16. $\sqrt{10}$ cm. 17. $9\frac{1}{3}$ cm. 18. 7.2 cm. 19. $\frac{l}{2\sin\left(45^\circ+\frac{\alpha}{4}\right)\cos\left(45^\circ-\frac{3\alpha}{4}\right)}$. 20. $\frac{2\sin(\alpha-30^\circ)}{\cos\alpha}$. 21. $\frac{\sqrt{1+8\cos^2\alpha}}{4\cos\alpha}$. 22. $\arccos\frac{1}{3}$, $\arccos\frac{\sqrt{3}}{3}$, and $\arccos\frac{\sqrt{3}}{3}$. 23. $\frac{2\sqrt{3(p^2+q^2+pq)}}{3}$. 24. Through the point E draw the straight line $EF \parallel AB$ (the point F lies on AC) and make necessary conclusions from considering the parallelogram $DBEF$. 25. $\frac{a}{2}(\sqrt{5}+1)$. 26. 1:2. Draw the midline in the triangle ADC. 27. $\frac{a\cos\alpha}{\sin\left(45^\circ+\frac{3\alpha}{2}\right)}$. 28. $\arcsin\frac{\sqrt{21}}{7}$ and $\arcsin\frac{\sqrt{21}}{14}$. 29. $\arctan\frac{1}{13}$. 30. $\arcsin\frac{72}{97}$. 31. 85°. From the point M drop the perpendiculars MK, ML, and MN to the sides AB, AC, and BC, respectively, set $\angle BMC=x$, $AM=k$, and express MN as a reference element in terms of k and x in two ways. 37. 40° and 25°. 38. 4 cm, 6 cm, and 8 cm, or $2\sqrt{6}$ cm, $4\sqrt{6}$ cm, and 6 cm; obtuse. 39. 0.75. 40. Right-angled. 41. $\sqrt{\frac{a^2+b^2}{5}}$. 42. $\sqrt{b(b+c)}$. 43. 10 cm. 44. $3\sqrt{5}$ cm. 10 cm, and 11 cm. 45. $\arcsin\sqrt{\frac{k}{2(k-1)}}$ if $k>2$, no solution if $1<k\leqslant 2$. 46. $\frac{\sqrt{a^2+b^2-2ab\cos\alpha}}{\sin\alpha}$. 47. 45°. 54. Consider the cases of a regular, an isosceles, and a right triangle. For an arbitrary triangle take advantage of the statement of Example 8 from Sec. 2. 55. Arrange

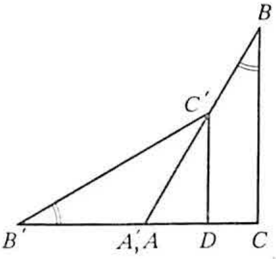


Fig. 164

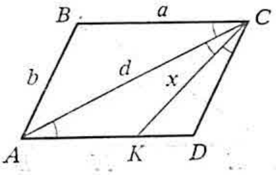


Fig. 165

the triangles in the way shown in Fig. 164 and draw $C'D \parallel BC$. Take advantage of the similarity of the triangles $AC'D$ and ABC. **56.** Denote the angles by x, $2x$, and $4x$ and, by means of the law of sines, express the larger sides of the triangle in terms of the smaller side. **57.** 90°, 22°30′, and 67°30′. Let CH, CD, and CM denote the altitude, angle bisector, and median of the triangle ABC, respectively. Setting $\angle C = 4x$ and $CH = h$, take advantage of the fact that $AH + MH = BH - MH$ and express all the elements of this equality in terms of h and x. **58.** $A+B=90°$ or $|A-B|=90°$. Express AD and BD in terms of the altitude h and the angles A and B. Consider two cases: A is an acute angle and A is an obtuse angle.

59. $\dfrac{2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}}{\sin\alpha}$. Through the point K draw the straight line $MP \parallel AC$ (point M lies on AB and point P on BC) and set $MK = KP = PC = a$. Express the line segment KC from the triangle KPC in terms of a, α, and β, and the line segment EK from the triangle MEK. **61.** Rectangle; $a-b$. **62.** $\sqrt{2n(m+n)}$ and $\sqrt{2(2m^2+3mn+n^2)}$. **63.** 17 cm, 10 cm, 21 cm, and $\sqrt{337}$ cm. **64.** $\dfrac{\sqrt{p^2+q^2+2pq\cos\alpha}}{\sin\alpha}$. **65.** $2\sqrt{2}\cos\dfrac{\pi(m-n)}{4(m+n)}$. **66.** $\dfrac{a\sin\alpha}{b+a\cos\alpha}$ and $\dfrac{b\sin\alpha}{a+b\cos\alpha}$. **67.** $\arccos\dfrac{7}{18}$. **68.** $\arcsin\dfrac{4-k^2}{k^2}$ and $\pi - \arcsin\dfrac{4-k^2}{k^2}$ if $\sqrt{2} \leqslant k < 2$; no solution if $k < \sqrt{2}$ or $k \geqslant 2$. **69.** Denote the sides of the parallelogram by a and ka, and the diagonals by d and kd. Using the formula relating the diagonals and sides, find the relation between a and d, and then make use of the law of cosines twice. **70.** Set $\angle APB = \alpha$, $\angle ADB = \beta$ and prove that $\tan(\alpha+\beta)=1$. **71.** $\sqrt{a^2+b^2+2b(\sqrt{a^2-b^2\sin^2\alpha}\cos\alpha+b\sin^2\alpha)}$. To find the obtuse angle between the smaller side and the smaller diagonal, take advantage of the law of sines, then find the diagonal, using the law of cosines

72. $\arccos\dfrac{(p^2+q^2)(n^2-m^2)}{2pq(n^2+m^2)}$ and $\pi - \arccos\dfrac{(p^2+q^2)(n^2-m^2)}{2pq(n^2+m^2)}$ Denote the sides of the parallelogram by px and qx, and the diagonals by my and ny. Take advantage of the formula relating the sides and diagonals of the parallelogram, then apply the law of cosines for expressing one of the diagonals in terms of the sides. **73.** $3\arccos\dfrac{2+k}{2k}$ and $\pi - 3\arccos\dfrac{2+k}{2k}$ if $k > 2$; no solution if $k \leqslant 2$. Let $BC = a$, $AB = b$, and $AC = d$ (Fig. 165). Draw the angle bisector CK in the triangle ADC. Express AK and CK from the isosceles triangle ACK in terms of α, and $\angle ACK = x$. Derive the rela-

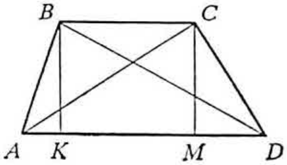


Fig. 166

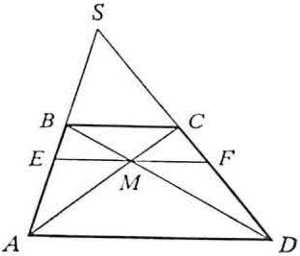


Fig. 167

tionship $\frac{a}{b}=2\cos x$ from the similarity of the triangles ABC and CKD. Applying the theorem on an angle bisector to the triangle ACD, derive the relationship $d=2a\cos x-b$. Adding the condition of the problem $d=\frac{2}{k}(a+b)$ to the obtained relationships, eliminate a, b, and d from the equalities and get the relationship $\cos x=\frac{k+2}{2k}$. 78. 15 cm. 79. 16 cm. 80. 2 cm. 81. $\frac{ac}{a+b}$, $\frac{ab}{a+b}$, and $\frac{bc}{a+b}$. 82. 14 cm, 12.5 cm, 29.4 cm, and 16.9 cm. 83. $h\cot\alpha$. 84. 1.5. 85. $\arctan\left(\frac{\sin\alpha}{2+\cos\alpha}\right)$. 86. $\frac{b^2+ab-c^2}{b}$. 87. From the regular triangle AOD establish that KD is perpendicular to AC and then $KP=0.5CD$. Draw a similar conclusion for MP. 88. $AC^2 = AD^2 + CD^2 - 2AD\cdot CD\cdot\cos D$ and $BD^2=AB^2+AD^2-2AB\cdot AD\cdot\cos A$ (Fig. 166). Add these equalities and, expressing AK and MD in terms of AB, CD, and the angles A and D, take into account that $MD+AK=AD-BC$. 89. Through the point M of intersection of the diagonals of the trapezoid draw the straight line $EF\parallel AD\parallel BC$ (Fig. 167). From the ratios $\frac{BE}{AB}=\frac{EM}{AD}=\frac{CF}{CD}=\frac{MF}{AD}$ establish that $EM=MF$. 90. $\frac{2ab}{a+b}$. See the hint to the preceding problem. 91. 40 cm. Prove that AKB and CED are right angles and that the median of the trapezoid lies on KE (see Problem 76). 92. $\frac{49\sqrt{2}}{2}$ cm. Find the line segment OK from the right triangle OCD. In the triangle OAB draw $OP\perp AP$ and, considering the angles thus formed, prove that $CM=AM$ and $OM=EM$. 96. $\frac{4ab}{a+b}$. 97. 9 cm, $9\sqrt{3}$ cm, and 18 cm. 98. $\frac{5}{4}\sqrt{m^2+n^2}$ and $\frac{5}{6}\sqrt{m^2+n^2}$. 99. $4\sqrt{2}$ cm and 18 cm. 100. $\arccos\frac{2\sqrt{5}}{5}$. 101. $2\sin\left(\alpha+\frac{\pi}{6}\right)$. 102. Take advantage of the facts that $PKLM$ is a parallelogram (P, K, L, and M are the midpoints of consecutive sides) and that $PELF$ is also a parallelogram (E and F are the midpoints of the diagonals). 103. Make use of the considerations from the solution of the preceding problem. 104. Suppose the contrary: one of the angles between the diagonals is acute, and the other is obtuse, and take advantage of Theorem 9 from Sec. 1. 105. Draw the diagonal and prove that its midpoint lies on the given line segment. 106. $2a\sqrt{7}$. 107. 150° and 90°.

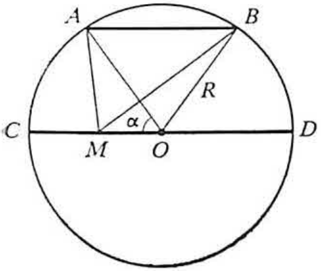


Fig. 168

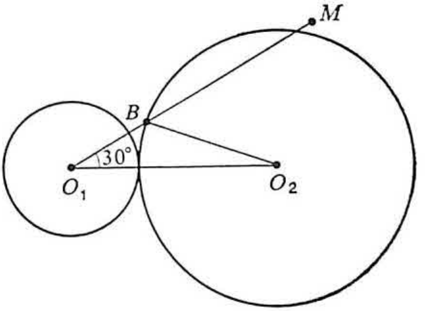


Fig. 169

108. $\arctan\frac{11}{3}$. Compute the tangents of the angles BAO, OAD, ODA, and ODC and, by means of them, find the tangent of the angle M. **109.** $\arctan\frac{1}{2}$. Introduce the following notation: $AB=2x$, $BC=3x$, and $BD=4\sqrt{2}x$. Express AD from the triangle ABD in terms of x, using the law of cosines, ascertain that the angle BAD is obtuse, find this angle by the law of sines. **110.** $\frac{m^2+m+1}{(m+1)^2}$. Denote $\angle BAE=\alpha$, $\angle KAD=\beta$, $\angle EAK=x$, $AD=a$, and $AB=ma$. Express the line segments BE and KD and then $\tan\alpha$ and $\tan\beta$ in terms of a and m, and take advantage of the fact that $\tan x=\cot(\alpha+\beta)$. **115.** 36°, 60°, 108°, and 156°. **116.** 24 cm. **117.** $\frac{a}{6}(3\pm\sqrt{3})$.

118. $2\sqrt{Rr}$. **119.** $\frac{\sqrt{14Rr-R^2-r^2}}{2\sqrt{3}}$. **120.** $14\pi+12\sqrt{3}$. **121.** 30 cm. **122.** $\frac{1+\sin\alpha}{1-\sin\alpha}$. **123.** $\frac{2a\sin\alpha\cos(\beta-\alpha)}{\sin\beta}$ if $\alpha<\beta$. **124.** Take advantage of the similarity of the triangles ABC and ABD. **125.** Draw the chord AP parallel to CD and take advantage of the fact that $AC=PD$. **126.** Set $OA=OB=R$, $\angle AOM=\alpha$ (Fig. 168) and, using the law of cosines, express AM from the triangle OAM and BM from the triangle BOM. **127.** $\frac{20}{3}$ cm and $\frac{15}{4}$ cm. Prove that $\angle BAC=90°$. Draw a common internal tangent. **128.** $\frac{R}{4}(8-3\sqrt{3}-\sqrt{7})$ and $\frac{R}{4}(3\sqrt{3}-\sqrt{7}-2)$. Set $O_1B=x$ and apply the law of cosines for O_2B in the triangle O_2O_1B (Fig. 169). **129.** $2\sqrt{b^2-((b-a)\cos\alpha-a)^2}$. Denote $O_2K=x$ (Fig. 170). Draw O_2P parallel to AB, and from the triangle O_2O_1P in which $\angle O_2O_1P=\alpha$ express x in terms of a, b, and α. **135.** 36°, 36°, and 108°. **136.** $\frac{2a(\sqrt{a^2+h^2}-a)^2}{h^2}$. **137.** 36 cm and 48 cm. **138.** 12π cm. **139.** $3\sqrt{5}$ cm. **140.** $\arccos\sqrt{\frac{2}{3}}$. **141.** $\frac{\sqrt{a^2+b^2-2ab\cos\gamma}}{2\sin\gamma}$. **142.** $b\tan^2\frac{\alpha}{2}$. **143.** If $k\leqslant 2$, then

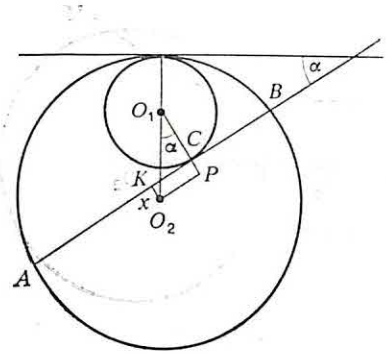


Fig. 170

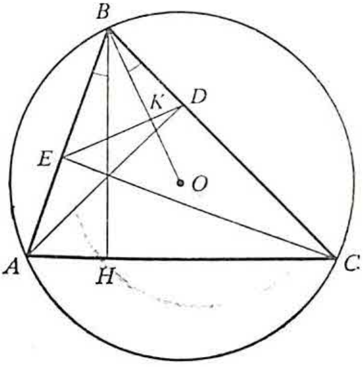


Fig. 171

$\arccos\frac{1-\sqrt{1-2k}}{2}$, $\arccos\frac{1-\sqrt{1-2k}}{2}$, and $\pi-2\arccos\frac{1-\sqrt{1-2k}}{2}$.
144. Applying the formulas $r=\frac{a+b-c}{2}$ and $h=\frac{ab}{c}$, transform the fraction $\frac{r}{h}$ to the form $\frac{\sin A+\cos A-1}{\sin 2A}$ Then take advantage of the fact that $\sin 2A \neq \sqrt{(\sin A+\cos A)^2-1}$. **145.** Prove that the orthocentre is symmetric with respect to some point of the circle about a side of the triangle. **146.** Construct on AM a point K such that $MK=BM$ and prove that the triangles ABK and BMC are congruent. **147.** Use the statement of the preceding problem. **148.** Denote the radius of the circle by R, $\angle ACK=\alpha$, and $\angle KCB=\beta$. Using the law of sines, express AK, KC, AB, and KB in terms of R, α, and β. **149.** Find the connection between the fractions on the left-hand side of the equality and the cotangents of the angles of the triangle. **150.** Let the circle be circumscribed about the triangle ABC, then the point M is taken on the arc AB, $MK\perp AC$, $MF\perp AB$, and $MD\perp BC$. Taking advantage of the fact that it is possible to circumscribe circles about the quadrilaterals $MAKF$ and $MFBD$, prove that $\angle AFK=\angle DFB$. **151.** Circumscribe a circle about the triangle ABC and extend the angle bisector CD to intersect the circle at the point P. Take advantage of the fact that the triangles BCD and ACP are similar and that $BD\cdot AD=CD\cdot DP$. **152.** Prove that $\angle ABH=\angle OBC$ (Fig. 171). Then take advantage of the facts that the triangles ABC and BDE are similar (D and E are the feet of the altitudes) and that the triangles BHA and BDK are also similar. **153.** $(3+\sqrt{3})$ cm. Take advantage of the fact that $CD\cdot AD=BD^2$ and apply the law of cosines to the triangle BAD.
154. $\frac{3R\sqrt{57}}{19}$. **155.** 120°. **156.** $2\arctan\frac{\sqrt{k^2+6k+1}-k-1}{2}$ and $\pi-2\arctan\frac{\sqrt{k^2+6k+1}-k-1}{2}$. **157.** $\arccos\frac{2}{3}$. **159.** 12 cm. **160.** $\frac{ac}{a+c}$.
161. $2\sqrt{3}$ cm, $4\sqrt{3}$ cm, and 6 cm. **162.** $\sqrt{a^2+ab+b^2}$. **163.** 2 cm. **164.** $2\sqrt{6}$ cm. **165.** 2 cm and 3 cm. **166.** 30°, 30°, and 120°. **167.** $a(\pi-\beta-\gamma)\times\frac{\sin\beta\sin\gamma}{\sin(\beta+\gamma)}$. **169.** 30°, 30° and 120°; $\frac{\sqrt{3}}{3}$ cm and $\frac{\sqrt{3}}{3}$ cm.
170. $35\frac{35}{47}$ cm. Determine the kind of the triangle. **171.** $\frac{c\sqrt{3}}{2}$. Let M be

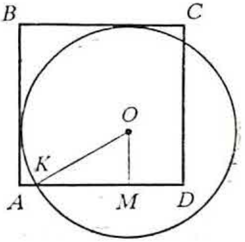


Fig. 172

the centroid, CM and DE intersect at P. Take advantage of the fact that $MP \cdot PC = PE \cdot DP$. **172.** $a^2+b^2=2c^2$. Take advantage of the result obtained in the preceding problem and apply the formula for expressing the median m_c in terms of the sides of the triangle. **173.** 2 cm and 4 cm. Take advantage of the fact that the perimeter of the triangle BDE is independent of the choice of the point of tangency and apply the law of cosines to the triangle BDE. **174.** $8\frac{1}{8}$ cm. Prove that the triangles DEC and ABC are similar, then $DE=EC=15$ cm. Take advantage of the facts that the centre of the circle lies at the point of intersection of AD and the perpendicular drawn to DE through its midpoint and that $\sin A=\frac{12}{13}$. **175.** $\beta b \cot \beta$. Take advantage of the facts that the line segment BH (H the orthocentre) is the diameter of the circle and that $BH=2OK$ (see Example 8 from Sec. 2), where O is the centre of the circle circumscribed about the triangle ABC, and OK is the perpendicular from the point O to AC. **177.** $10\frac{5}{8}$ cm. **178.** $\dfrac{\sqrt{a^2+b^2+2ab\cos^2\alpha}}{2\sin 2\alpha}$. **179.** $\dfrac{8R}{5}$. **180.** $2\cos^2\dfrac{\alpha}{4}$. **181.** $\arccos\dfrac{k-1}{k}$, $\pi-\arccos\dfrac{k-1}{k}$ if $k\geqslant 1$. **182.** $\dfrac{4\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{\pi\sin\alpha\sin\beta}$. **183.** (a) Let one of the diagonals form angles α, β, γ, and δ with the sides. Express a and b, c and m, d_1 and d_2 in terms of the radius of the circle, and the angles α, β, γ, and δ by the law of sines. **184.** Let the altitudes, BD, AF, and CE intersect at the point H. Circumscribe a circle about the quadrilateral $AEHD$ and prove that $AC \cdot CD = CE \cdot CH = ab\cos C$. In a similar way, prove that $BD \cdot BH = ac\cos B$, $AF \cdot AH = bc\cos A$ and then apply the law of cosines to each of the sides a, b, and c. **185.** 15 cm and 20 cm. Circumscribe a circle about the given triangle. **186.** 17 cm. Apply the Pythagorean theorem to the triangle OKM (Fig. 172). **187.** 2 cm and 2 cm. Take advantage of the fact that the perimeter of the triangle AMP is independent of the choice of the point of tangency and apply the law of cosines to the triangle AMP. **188.** $\arcsin\dfrac{1}{k}\sqrt{\dfrac{1+\sqrt{1+4k^2}}{2}}$ if $k\geqslant\sqrt{2}$. Denote the radius of the inscribed circle by r, and the acute angle of the trapezoid by x. Express the sides of the trapezoid, the diagonal, and then the radius of the circumscribed circle in terms of r and x. **189.** $2\sqrt{4\tan^2\alpha+3}$. Prove that EM is a median of the triangle CED, and, therefore, $EM=\frac{1}{2}CD$. **190.** $\dfrac{30\tan\alpha}{\sqrt{25+36\tan^2\alpha}}$. Prove that EH is perpendicular to AB. **191.** $\sqrt{\dfrac{65}{2}}$. Prove that $\sin\angle(A+B)<0$. This means that the straight lines AD and BC intersect at the point K lying to the left of AB (Fig. 173), and the circle in question is inscribed in the triangle KCD. Find the angles of the triangle ODC. **193.** $12\sqrt{5}$ cm. **194.** $2\frac{46}{49}$ cm. **195.** $\sqrt{10}$ cm.

196. $\frac{R}{2}(\sqrt{7}-1)$. 197. $\frac{\sin\alpha}{1+\sin\alpha}$. 198. $\frac{R\sin\frac{\alpha}{2}}{\sin\left(\frac{\alpha}{2}+\frac{\pi}{6}\right)}$. 199. $\frac{2}{5}R\sqrt{4+\sin^2\alpha}-\frac{4}{5}R\cos\alpha$. 200. $\frac{4R\sqrt{3}\sin^2\frac{\alpha}{2}}{3}$. 201. $\frac{R}{2}(\sqrt{3}\cos\alpha+\sqrt{3+\sin^2\alpha})$.

202. $\frac{2a\sqrt{ab}}{b}$. 203. 3 cm. 204. $\frac{b}{2}\tan^3\frac{\alpha}{2}$ 205. $4R\sin^2\frac{\alpha}{8}\cos\frac{\alpha}{4}$. 206. $0.6a$.

207. $\frac{6-4\sqrt{2}}{3}$ cm. 208. $\frac{a\sqrt{3}+r-\sqrt{4r^2+2ar\sqrt{3}}}{3}$, $r\geqslant\frac{\sqrt{3}}{2}a$.

209. $\frac{b}{2}\tan\frac{\alpha}{2}\tan^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$. 210. $\frac{b}{4\sin\alpha\cos^2\frac{\alpha}{2}}$. 211. $4R\cos\frac{\alpha}{4}\sin^2\frac{\alpha}{8}$.

212. Let the circles circumscribed about the triangles ADM and BDK intersect at the point P. Bearing in mind that $\angle DAM+\angle DPM=180°$ and $\angle DBK+\angle DPK=180°$, prove that $\angle MCK+\angle MPK=180°$. 213. 120°. Express the radius of the inscribed circle, as a reference element, in terms of the known linear elements and $\angle AOB=2x$ from the two triangles O_1PH and O_1MB, where P is the midpoint of KH, and M is the midpoint of AB. 214. $R\left(\sqrt{1-\frac{1}{3}\cos^2\frac{\alpha}{2}}-\cos\frac{\alpha}{2}\right)$ if $\alpha\geqslant\frac{\pi}{3}$. Express OC in terms of R and α and apply the law of cosines to the triangle ODC (Fig. 174) in which $OD=R$, $CD=x$, and $\angle OCD=150°$. 215. $2r\tan\frac{\alpha}{2}\left(1-\tan\frac{\alpha}{2}\right)$. Apply the law of cosines to the triangle OAO_1, where O and O_1 are the centres of the circles. 216. $\tan^2\frac{\alpha}{2}\left(\frac{1}{\sin\frac{\alpha}{2}}\sqrt{r^2+\arcsin\alpha}-r-a\cot\frac{\alpha}{2}\right)$. See the hint to the preceding problem. 217. $\frac{\sin\frac{\alpha}{2}\left(b\cos\frac{\alpha}{2}-\sqrt{4r^2-b^2}\sin\frac{\alpha}{2}-2r\sin\frac{\alpha}{2}\right)}{\cos^2\frac{\alpha}{2}}$. Find the cosine of the angle OAO_1 and apply the law of cosines to the triangle OAO_1 (O and

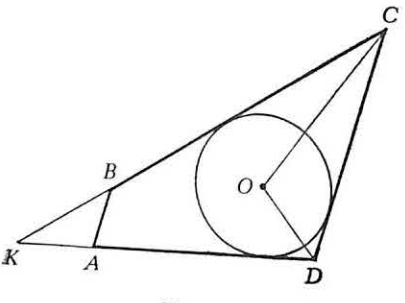


Fig. 173

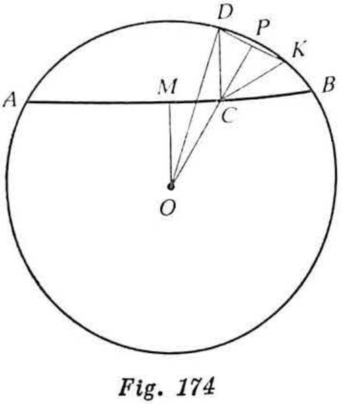


Fig. 174

O_1 are the centres of the circles). **218.** $\dfrac{b+c-2\sqrt{bc}\cos\alpha}{2\sin^2\alpha}$. Take advantage of the facts that $AM^2=AC\cdot AB$ (Fig. 175) and that $AM=AD+DM$. **219.** $DE=\dfrac{b^2-a^2}{b}$ and $R=\dfrac{a^2+b^2-2ab\cos\alpha}{2b\sin\alpha}$. Take advantage of the fact that $OC^2=OD\cdot OE$. Apply the laws of sines and cosines to the triangles OEC, ODC, and CED. **223.** 360 cm². **224.** (a) 9 cm²; (b) 3 cm²; (c) 12 cm². **225.** 4 cm². **226.** $\dfrac{a^2}{2}(3+2\sqrt{2})$. **227.** $\dfrac{180\sqrt{3}}{19}$. **228.** $\dfrac{1}{4}c^2(q^2-1)$ if $1<q\leqslant\sqrt{2}$. **229.** $m^2\cos^2\dfrac{\alpha}{2}\cot\alpha$. **230.** $m\sin\beta\,(\sqrt{c^2-m^2\sin^2\beta}+m\cos\beta)$.

231. $\dfrac{\sin(\alpha-\beta)}{2\cos\alpha\sin\beta}$. **232.** $\dfrac{2\sqrt{3}\cos\alpha+\sin\alpha}{\sin\alpha}$. **233.** (a) $\dfrac{\tan\dfrac{|\alpha-\gamma|}{2}}{2\tan\dfrac{\alpha+\gamma}{2}}$;

(b) $\dfrac{\sin(\alpha-\gamma)}{2\sin(\alpha+\gamma)}$; (c) $\dfrac{\tan\dfrac{|\alpha-\gamma|}{2}\sin\alpha\sin\gamma}{\sin(\alpha+\gamma)}$.

234. $\dfrac{l(a+b)}{4ab}\sqrt{4a^2b^2-l^2(a+b)^2}$. **235.** $\dfrac{9\sqrt{3}}{4}$. **236.** Express the sides of the triangle in terms of R and the angles A, B, and C (using the law of sines) and prove that $\sin A\sin B\sin C\leqslant\dfrac{3}{4}$. **237.** $ab\sqrt{3}$. Express the area of the triangle in two ways: by Hero's formula and the formula $S=pr$. **238.** $(\sqrt{S_1}+\sqrt{S_2})^2$. Introduce the notation: $BQ=x$, $CQ=y$ and take advantage of the facts that $\dfrac{S_{ABC}}{S_2}=\left(\dfrac{x+y}{y}\right)^2$ and that $\dfrac{S_1}{S_2}=\dfrac{x^2}{y^2}$. **239.** $(\sqrt{S_1}+\sqrt{S_2}+\sqrt{S_3})^2$. See the hint to the preceding problem. **240.** $\dfrac{720}{17}$. Determine the kind of the triangle. **241.** 200 cm². Let AK and CM be the perpendiculars to the tangent, and BD the altitude of the triangle ABC. Establish from the similarity of the triangles ABD and BMC that $\dfrac{BD}{16}=\dfrac{AB}{BC}$, and from the similarity of the triangles AKB and BDC that $\dfrac{BD}{25}=\dfrac{BC}{AB}$. **242.** $\dfrac{abc}{mkc+nkb+mna}$. Taking advantage of the fact that $S_{DEF}=S_{MDE}+S_{MEF}+S_{MDF}$, express S_{DEF} in terms of m, n, and k and the sines of the angles of the triangle ABC. Then express the sines of the angles of the triangle ABC in terms of its area and sides. **243.** $2\cos\alpha\cos\beta\cos\gamma$. Taking advantage of the fact that the triangles ABC and BFD are similar (see Example 11 from Sec. 3), ascertain that $FD=b\cos\beta$; analogously, $DE=c\cos\gamma$. Taking advantage of the fact that $\angle ABE=\angle FDA=\angle ADE=\angle FCA$ (see Example 3 from Sec. 1), express the angle FDE in terms of the angle BAC. **244.** $\dfrac{3\sqrt{2}}{4}$. Introduce the notation:

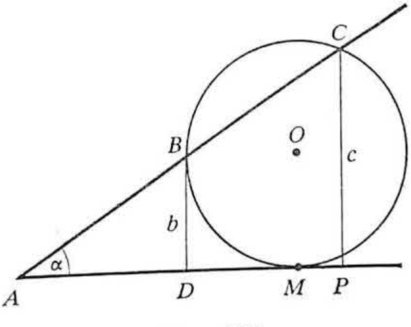


Fig. 175

$AC=b$ and $BC=a$. Applying the law of cosines to the triangles ABC, ACD, and BCD, get the system: $\begin{cases} 2a^2+b^2=12, \\ a^2+b^2+ab=9. \end{cases}$ **245.** $3\sqrt{2}$. See the hint to the preceding problem. **246.** $\frac{27}{65}$ cm². See Example 5 from Sec. 4. **247.** 210 cm². Let $OP \perp AC$ ($OP=10$ cm), BH be the altitude of the triangle ABC, and $OK \perp BH$. From the triangle BHD find that $BH=20$ cm. Then in the triangle OBK, $BK=10$ cm and $\angle OBK=\angle BDH$; from this triangle find the radius of the circle, then AP, AC, and, using the formula $CD \cdot AD=BD^2$, find CD. **249.** $\frac{(a-b)^2 \sin\alpha}{2}$. **250.** a^2. **251.** 135 cm². **252.** 48 cm². **253.** $\frac{5R^2\sqrt{3}}{4}$. **254.** 147 cm². **255.** 126 cm². **256.** 1476 cm². **257.** 336 cm². **258.** 168 cm². **259.** $25\sqrt{3}$ cm². **260.** 150 cm². **261.** $\frac{1}{2}(a^2-b^2)\tan\alpha$. **262.** $2R^2 \sin 2\alpha \sin^2\alpha$. **263.** $\frac{4\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{\pi\sin\alpha\sin\beta}$. **264.** $\frac{H^2}{2}\sin\beta\cos(\alpha-\gamma)$. **265.** Draw the altitude BH and bring into coincidence the point M with the point H. **266.** $(a+b)^2$. Prove that $S_{AOB}=S_{COD}$ and take advantage of the fact that $\frac{S_{BOC}}{S_{COD}}=\frac{BO}{OD}=\frac{b}{a}$. **267.** 20 cm². The straight lines divide the rhombus into nine parts, four of which are triangles. Denote the area of the small triangle by x, show that the area of the quadrilateral adjacent to the side is equal to $3x$, and express the area of the triangle ABN in terms of the area of the rhombus. **268.** $\frac{8ab\sqrt{ab}}{a+b}$. See Example 7 from Sec. 3. **269.** 70 cm². Take advantage of the fact that $\frac{S_{AOD}}{S_{COD}}=\frac{AO}{OC}$. **270.** 40 cm². Denote KD by x (P and M are the points of contact of the circle with AD and AB, respectively). Taking advantage of the fact that $PD^2=CD \cdot KD$, express the line segments PD, OP, and MK in terms of x. **271.** 20 cm² and 10.4 cm². Introduce the notation: $AB=x$, $BC=2x$, and $\angle ABM=\alpha$. Apply the law of cosines to AM (in the triangle ABM) and CM (in the triangle BMC), and then use the formula $\sin^2\alpha+\cos^2\alpha=1$. **272.** $4S+2c^2$. **273.** 926 cm². **274.** $2a^2(\sqrt{2}-1)$. **275.** $\frac{3}{2}a^2$. **276.** $289(\sqrt{3}+1)$ cm². **277.** $\frac{R^2}{4}(8\sqrt{3}-9)$. **278.** $\frac{37}{64}$. **279.** 11 cm². Introduce the notation: $BF=x$, $AF=2x$, $BK=y$, $CK=3y$, and compare the areas of the triangles BKF and ABC. **280.** $\frac{784}{7225}$. **281.** $\frac{a^2}{24}(3\sqrt{3}-\pi)$. **282.** $\frac{5\sqrt{3}\pi-18}{54}$. **283.** (a) $(a+b)\sqrt{ab}-\frac{\pi b^2}{2}-\frac{1}{2}(a^2-b^2)\arccos\frac{a-b}{a+b}$; (b) $\frac{\pi a^2 b^2}{(\sqrt{a}+\sqrt{b})^4}$. **284.** $\frac{a^2}{18}(3\sqrt{3}-\pi)$. **285.** $\frac{a^2}{2}(\pi-2)$. **286.** (a) $\frac{R^2}{6}(7-4\sqrt{3})(2\sqrt{3}-\pi)$; (b) $R^2(3-2\sqrt{2})(4-\pi)$; (c) $\frac{2R^2}{9}(3\sqrt{3}-\pi)$. **287.** $R^2(\alpha+\sin\alpha)$. **288.** $\frac{\pi a^2}{6}$. **289.** $10R^2\left(\sqrt{3}+\frac{2\pi}{3}\right)$. **290.** $\frac{\pi a^2}{18}(2-\sqrt{3})$. **291.** $\frac{(4\pi-3\sqrt{3})(4+\sqrt{7})}{27}$. **292.** Prove that $CD^2=AC \cdot BC$.

293. $\frac{b^2}{4}\cot\beta\,(\beta-\sin\beta\cos(2\alpha+\beta))$. Take advantage of the fact that the circle passes through the point C. **294.** $\frac{na^2}{1+\tan^2\alpha}\left(\cot\alpha+\pi-\frac{\pi n}{2}\right)$, where $\alpha=\frac{\pi}{n}$. The "star" is the difference between the n-gon with vertices at the centres of the circles and n sectors. **295.** $\frac{na^2\cos^2\alpha}{4(1+\sin\alpha)^2}\left(\cot\alpha+\pi-\frac{\pi n}{2}\right)$, where $\alpha=\frac{\pi}{n}$. See the hint to the preceding problem. **301.** 75 cm. **302.** $2\sqrt{\frac{S}{3}\cot\frac{\alpha}{2}}$.

303. (a) $\frac{2ab\cos\frac{\gamma}{2}}{a+b}$; (b) $\frac{ab\sin\gamma}{c}$. **304.** $\sqrt{\frac{a^2\sin\alpha+2ah\cos\alpha+2ah}{\sin\alpha}}$. **305.** $\sqrt{\frac{2}{4-\pi}}$. **306.** $\arctan\frac{a^2-b^2}{4S}$. **307.** $\frac{\sqrt{2S\sin\alpha\sin\beta\sin\gamma}}{\sin\alpha}$. **308.** 120° and $\sqrt{3}$ cm. **309.** $\frac{7\sqrt{145}}{5}$ cm. **310.** Introduce the notation: $b=a+d$ and $c=a+2d$, express the area S in terms of a and d, using Hero's formula, and then take advantage of the formulas $R=\frac{abc}{4S}$ and $r=\frac{S}{p}$. **311.** $\sqrt{\frac{a^2+b^2}{2}}$. Let the altitude of the trapezoid be divided by the line segment x into the parts h and kh $(k>1)$. Then $\frac{b+x}{2}kh=\frac{1}{2}\frac{a+b}{2}(k+1)h$ and $\frac{x-b}{a-b}=\frac{k}{k+1}$. From this system find x. **312.** 14.4 cm. Take advantage of the similarity of the triangles AKD and CKM, where K is the point of intersection of AM and CD. **313.** 4.5 cm. See Example 11 from Sec. 3. **314.** 24 cm. See Example 11 from Sec. 3. **315.** 4 cm. See Example 4 from Sec. 4. **316.** $\frac{2\sqrt{S_2(S_1+S_2)}}{\sqrt[4]{4S_1^2-S_2^2}}$. Introduce the notation: $AC=x$ and $AB=y$. Express the area of the triangle ABC in terms of x and y. Prove that $\frac{y}{x}=\frac{S_1}{S_2}$. **317.** 8.25 cm. Find the radius of the circumscribed circle and take advantage of the statement of Example 8 from Sec. 2. **318.** $\frac{8\sqrt{3}}{3}$ cm, $\frac{26\sqrt{3}}{3}$ cm, and $10\sqrt{3}$ cm. Prove that $a:b:c=4:13:15$ and introduce the notation: $a=4x$, $b=13x$, and $c=15x$. **319.** $\frac{2\cos\frac{\gamma}{3}+3}{1+6\cos\frac{\gamma}{3}}$. Introduce an auxiliary parameter $AC=b$, then $BC=3b$. Apply the method of areas $S_{ACD}+S_{DCB}=S_{ACK}+S_{KCB}$. **320.** $\frac{\sqrt{7}}{9}(4+\sqrt{7})$. See Example 13 from Sec. 4. **321.** $\frac{33+12\sqrt{6}}{25}$. See Example 13 from Sec. 4. **323.** The triangles $A_1B_1C_1$ and PQR are symmetric with respect to the point M. **324.** Consider the symmetry with respect to the midpoint of the side whose length is unknown, then the triangle with sides a, b, and $2m$ can be constructed,

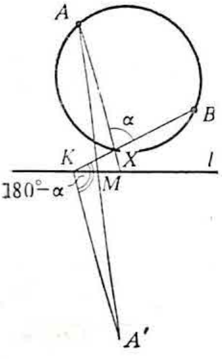


Fig. 176

$\frac{a-b}{2} < m < \frac{a+b}{2}$, where m is the length of the given median. **325.** Ascertain that the point of intersection of the medians of the triangle is the centre of symmetry carrying the triangle ABC into the constructed triangle. **326.** Consider the symmetry with respect to the point P. **327.** Take advantage of the fact that the point of intersection of the diagonals of a parallelogram is its centre of symmetry. **328.** Draw a straight line through the centre of symmetry of the parallelogram. **329.** Take advantage of the facts that the centre of the circle is its centre of symmetry and that the corresponding lines in central symmetry are parallel. **330.** The centre of the circle is the centre of symmetry of the circumscribed hexagon. **331.** Ascertain that the point O of intersection of the diagonals AD and BE is the centre of symmetry of the hexagon $ABCDEF$. Further, $S_{\Delta COA} = S_{\Delta OAF}$, $S_{\Delta COE} = S_{\Delta OEF}$, and $S_{\Delta EOA} = S_{\Delta ODE}$. Adding these equalities termwise, find that $S_{\Delta ACE} = \frac{1}{2} S_{ABCDEF}$. **332.** Let the point A' be symmetric with respect to the point A, M being the centre of symmetry (Fig. 176). Then the angle BKA' is known (K is the point of intersection of straight lines BX and l). Hence, on the line segment $A'B$, where $A' = Z_M(A)$, construct a segment containing the angle equal to $180° - \alpha$, K being the point of intersection of the arc of the segment and the straight line l. The problem has two solutions. **333.** The desired straight line m is passed through the points symmetric with respect to the point M and belonging to the sides of the angle ABC. To prove this fact, establish that the area of the triangle cut off by a certain straight line l containing the point M and different from the straight line m is greater than the area of the triangle cut off by the straight line m. **334.** Take advantage of the symmetry whose centre coincides with the centre of the circle. **335.** The midpoint of the side BC (point M) is the centre of symmetry of the parallelogram $BXCY$ (Fig. 177). Therefore, the transformation carrying x into y is a symmetry with respect to the centre M, or, which is the same, a homothetic transformation H_1 with centre M and the ratio of similitude $k_1 = -1$. Since $\overrightarrow{XY} = \overrightarrow{AZ}$, $\overrightarrow{AZ} = 2\overrightarrow{MY}$. Consequently, the transformation carrying Y into Z is a homothetic transformation H_2 with centre S at the point of intersection of the straight lines ZY and AM and the ratio of similitude $k_2 = 2$. The composition of these homothetic transformations $H_2 \circ H_1$ is a certain homothetic transformation H with the ratio of similitude $k = k_1 \cdot k_2 = -2$. Let us find its centre O. Since $H_1(M) = M$, $H(M) = H_2(M) \circ H_1(M) = H_2(M) = A$. Consequently, $\overrightarrow{OA} = -2\overrightarrow{OM}$. This means that O is the point of intersection of the medians of the triangle ABC. Thus, $x \to z$ is a homothetic transformation with centre at the point of intersection of the medians of the triangle ABC and the ratio of similitude $k =$ -2. **336.** (a) Let $ABCD$ be the desired parallelogram inscribed in the given quadrilateral $LMNK$, O be the centre of symmetry of the parallelogram, and the points B and D belong to MN and KL, respectively (Fig. 178). Obviously, A is the point of intersection of OML and the image of MK in the symmetry with respect to the centre O, and C is the point of intersection of MK and the image of ML in the symmetry with respect to the centre O. Let the point A belong to LM, and B to MN. Since $T_{\overrightarrow{AB}}(A) = B$ and $T_{\overrightarrow{AB}}(D) = C$, this means that C is the point of intersection of KM and the image of ML.

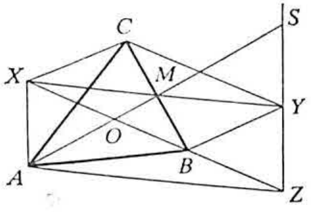


Fig. 177

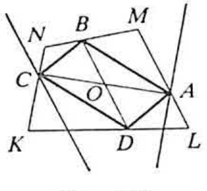


Fig. 178

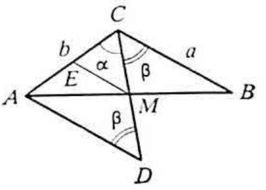


Fig. 179

Remark. In this problem it is supposed that the vertices of the desired parallelogram may belong to the sides of the quadrilateral or to their extensions. **337.** Let us denote $BC = a$ and $AC = b$, then, by the hypothesis, $a > b$. Let us dwell on two methods of solution of this problem. *First Method.* Construct the point D symmetric with respect to the point C, M being the centre of symmetry (the midpoint of the side AB), and join the point D to the vertex A (Fig. 179). The constructed triangle ADM is symmetric with respect to the triangle BCM, M being the centre of symmetry. Consequently, $AD = BC = a$ and $\angle BCM = \angle ADM = \beta$. Thus, the triangle ACD is obtained, which contains the angles α and β we are interested in, the opposite sides AD and AC being respectively equal to a and b. According to the hypothesis, $a > b$, and, therefore, $\alpha > \beta$. *Second Method.* Join the midpoint E of the side AC to the point M to form the triangle CEM with sides $EM = \frac{a}{2}$ and $CE = \frac{b}{2}$. The angles of the triangle CEM, which are opposite to these sides, are equal to α and β, respectively. This can be readily proved, using the midline of the triangle. **338.** (a) Take one vertex on the axis of symmetry, two others not on the axis and construct the vertices symmetric to them; (b) a pentagon having two axes of symmetry is a regular one. **339.** Take advantage of axial symmetry, carrying one straight line into the other. **340.** Construct the image of the smaller side, using the symmetry with respect to the bisector of the angle opposite to the given side. **341.** After having used the symmetry with respect to the middle perpendicular drawn to the third side, the problem is reduced to constructing a triangle given two sides and the included angle. **343.** See Fig. 180. **344.** It is possible to carry out the proof by contradiction. If a straight line containing the diagonal of the pentagon serves as its axis of symmetry, then two and only two vertices of the polygon lie on this axis (since the pentagon is convex). The remaining three vertices must be equally distributed on each of the half-planes with boundary l. But it is impossible for three vertices. Consequently, such a pentagon does not exist. **345.** The solution is similar to that of the preceding problem. **346.** Construct an auxiliary triangle given two sides and the included angle and then use the symmetry with respect to the middle perpendicular to the third side. Either the given angle or that adjacent to it is taken. **347.** See the hint to Problem 346. **348.** (a) Construct any diameter of one circle and a perpendicular diameter of the other circle; (b) extend the diameter AB of the smaller circle to intersect the greater circle at the point C. Construct the axes of symmetry of the line segments AC and BC. **349.** The composition of axial symmetries $S_r \circ S_q \circ S_p$ is a symmetry with axis l and $S_r \circ S_q \circ S_p(A) = A$. Consequently, $S_l(A) = A$, that is, A belongs to l. Construct the straight line l, taking into account that $\angle(p, q) = \angle(l, r)$. Any point of the straight line l can serve as the vertex A. **350.** Through the centre of the circle draw three straight lines perpendicular to the given ones. Take advantage of the result obtained in the preceding problem. **351.** Let the straight line DH intersect the straight line CO at the point H', and let C' be diametrically opposite to C (Fig. 181). The rays CH and CO are symmetric about CM,

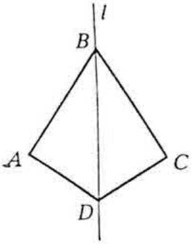


Fig. 180

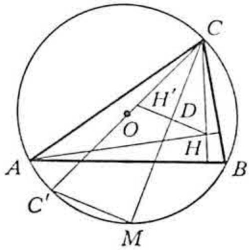


Fig. 181

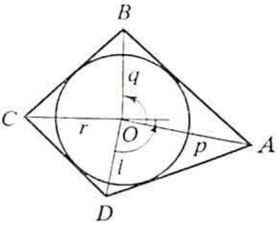


Fig. 182

and $CD:CM = CH':CC' = CH:CC' = 2R \cos C:2R = \cos C$. 352. If $OM = p$,
$ON = q$, $OP = r$, and $OQ = l$, then $\sigma = S_l \circ S_r \circ S_q \circ S_p$ is an identical mo-
tion since $\sigma(0) = 0$ and $\sigma(A) = A$. Consequently, $S_q \circ S_p = S_r \circ S_l$ or
$\angle(p, q) = \angle(l, r)$. 353. Consider the composition $\delta = S_l \circ S_r \circ S_q \circ S_p$, where p,
q, r, and l are the straight lines containing the bisectors of the angles of the quad-
rilaterals. Since $\delta(AD) = AD$ and $\delta(0) = 0$, δ is an identical transformation,
$S_q \circ S_p = S_r \circ S_l$, and, hence, $\angle(p, q) = \angle(l, r)$. Consequently, $\angle COD +$
$\angle AOB = 180°$ (Fig. 182). 356. Denote the bases of the trapezoid by BC and AD,
and the axis of symmetry by l. Then $S_l(A) = D$, $S_l(B) = C$, and $S_l(AB) =$
DC. Hence, $S_p(AB) = DC$ and $S_l(AC) = DB$. But the point of intersection of
a straight line and its image in axial symmetry belongs to the axis. Consequently,
the point of intersection of the straight lines AB and DC belongs to l and the
point of intersection of the line segments BC and AC also belongs to l. 357. Let
O be the centre of the circle, AB and CD parallel chords of this circle, M
the midpoint of the chord AB, and N the midpoint of the chord CD. Since $AO =$
OB and $AM = MB$, OM is the axis of symmetry of the points A and B, whence
it follows that $OM \perp AB$. Similarly, ON is the axis of symmetry of the points C
and D, and $ON \perp CD$. Taking into account that $AB \parallel CD$, find that $OM \parallel ON$,
and, therefore, $OM = ON$. 358. Let O be the centre of the circle F_1, and O_1 the
centre of the circles F_2 and F_3. The circles F_1 and F_2 intersect at A and B, and
the circles F_1 and F_3 at C and D. Then OO_1 is the axis of symmetry of the figure,
which is the intersection of the circles F_1, F_2, and F_3. 359. Let M be a common
point of the given circles ω_1, ω_2, and ω_3. The points A, B, and C are second points
of intersection of ω_2 and ω_3, ω_3 and ω_1, ω_1 and ω_2. Consider the composition of
three axial symmetries, i.e. S_{MA}, S_{MB}, and S_{MC}, which is a symmetry with
axis MO_2, where O_2 is the centre of the circle ω_2. If the point P is diametrically
opposite to the point M on ω_2, then $S_{MA}(P) = Q$, $S_{MB}(Q) = R$, and $S_{MC}(R) =$
P, and $PA = AQ$, $QB = BR$, and $RC = CP$. Consequently, AB, BC, and CA
are the midlines in the triangle PQR, and, therefore, the circle circumscribed
about the triangle ABC has the same radius as the circle circumscribed about
the triangle QAB. 360. See the solution of Problem 359. 361. Let A be the given
point, l the given straight line, and $AO \perp l$. It is easy to see that the desired locus
F is symmetric about the straight line AO, and, therefore, it is sufficient to con-
sider only the half-plane to the right of AO (including this straight line). Ob-
viously, a point K (K belongs to AO) such that $AK = 2KO$ belongs to the locus
F. Through the point K draw a straight line m parallel to the straight line l.
Let M be the centre of the regular triangle ABC and lie above the straight line
m, and let AD be the altitude of the triangle ABC. Since AOB and ADB are
right angles, the points O and D lie on the circle of diameter AB, and, therefore,
$\angle AOD = \angle ABD = 60°$. But the triangles AOD and AKM are similar (their
analogous sides are proportional), and, hence, $\angle AKM = 60°$. If the point M
lies below the straight line m, then find analogously that $\angle AKM = 120°$. The
converse is also true: any point M such that the angle AKM is equal either to

60° or to 120° belongs to the required locus F. Finally, get that the locus F represents two straight lines passing through the point K at angles of 60° and 120° to the straight line AO, or, which is the same, at angles of 30° and 150° to the given straight line l. **362.** Let A be the given point, l the given straight line, A' the point symmetric with respect to A about l, and m the straight line passing through A' and parallel to l. It is easy to see that the sought-for locus F of points is symmetric about the straight line AA', and, therefore, consider only the half-plane lying to the right of AA' (excluding this straight line). Let C be a point belonging to the locus F and lying above the straight line m, and let B be the second vertex of the regular triangle ABC. Then the circle of radius AB with centre at B is passed through the points A, A', and C. Hence it follows at once that $\angle AA'C = 30^\circ$. And if the point C lies below the straight line m, then $\angle AA'C = 150^\circ$. It is also clear that there is only one point A' belonging to the locus F on the straight line m. Thus, the points of the locus F, which are situated in the half-plane to the right of AA', lie on two rays emanating from the point A' and making angles of 30° and 150° with AA'. Prove that this pair of rays is the right-hand half of the sought-for set of points F. Finally, get that the locus F represents two straight lines passing through the point A' at angles of 30° and 150° to the straight line AA', or, which is the same, at angles of 60° and 120° to the given straight line l. **363.** Construct the point M' corresponding to the point M after rotating it about O through an angle of 90°. The distance from the point O to $M'N$ is equal to half the length of the side of the square. **364.** Construct the image of one of the circles after rotating it about the point O, the centre of rotation through an angle of 60°. The point of intersection of the second of the given circles and the constructed one is the second vertex of the triangle. **365.** Construct the chord of the circle of the given length and the circle concentric to the given one and passing through the given point. Consider the rotation about the centre of the circles in which the point of intersection of the constructed chord and circle corresponds to the given point. **366.** The rotation about the centre of the triangle through an angle of 120° carries M into N, N into P, and P into M; $MN = \frac{a\sqrt{3}}{3}$. **367.** The rotation about the centre of the square through an angle of 90° carries P into Q, Q into R, R into S, and S into P; $PQ = \frac{a\sqrt{10}}{4}$. **370.** The line segments AD and AE can be laid off in two ways. In one case, the condition of the problem is not fulfilled. In the other case, there is a circle passing through the points B, C, D, and E (its centre lies on the axis of symmetry l of the "butterfly" $DEABC$). **371.** Let O be the point of intersection of the diagonals AC and BD of the square $ABCD$, MN and KL the intersections of the square and the given straight lines (the points M, N, K, and L belong respectively to the sides AB, CD, BC, and AD of the square). Then $R_O^{90^\circ}(AB) = AD$. The image of the point M is a point M' of the line segment AD such that $\angle MOM' = 90^\circ$, that is, the point L. Analogously, $R_O^{90^\circ}(N) = K$. Consequently, $R_O^{90^\circ}(MN) = KL$, and, therefore, $MN = KL$. **372.** To prove the equality of the line segments, it is necessary to find a motion in which one of the segments goes into the other. Since the angle between the straight lines containing the indicated line segments is equal to 60°, it is only natural to consider the rotation about the point O. Since the rotation about the point O through an angle of 120° carries the triangle into itself, it is appropriate to consider the rotation about O through 120°. In this rotation, the point A goes into B, B into C, C into A, AB into BC, BC into CA, and CA into AB. The point E belonging to AC goes into the point M belonging to AB (Fig. 183); the point F belonging to AB goes into the point N belonging to BC ($\angle EOM = 120^\circ$ and $\angle FON = 120^\circ$). Consequently, EF goes into MN. Hence, $EF = MN$. **373.** Let PKL be the desired triangle (Fig. 184). Then the points K and L are equidistant from the point P, belong to the straight lines a and b, respectively, and are seen from the point P at an angle of 60°. Since

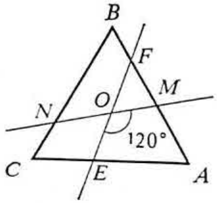


Fig. 183

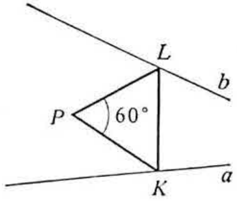


Fig. 184

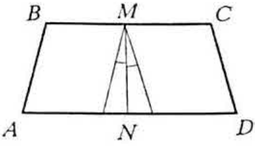


Fig. 185

the point L is the image of the point K when rotating the point P through an angle of 60°, it belongs to the image of the straight line a in the indicated rotation (that is, the point L is a common point of the straight line $a' = R_P^{60°}(a)$ and the straight line b). The point K is the preimage of the point L. If $b = R_P^{60°}(a)$, then the problem has infinitely many solutions. In the remaining cases, the problem has no more than two solutions, since the straight line b has no more than one point of intersection with the straight line a' and no more than one point of intersection with the straight line $a'' = R_{\mathbf{P}}^{-60°}(a)$. **374.** Perpendicularity of two straight lines will be proved if one of these lines is carried into the other by rotating it through an angle of 90°. Analysing the conditions of the problem, note that the points M and B are equidistant from the point A, and $\angle MAB =$ 90°. Analogously, $AC = AP$, and $\angle CAP = 90°$. Hence, the rotation about the point A through an angle of 90° clockwise carries the point M into the point B, and the point C into the point P. **375.** Consider the rotation of the plane about the point M through an angle of 90°. **376.** Consider the composition of two rotations, $R_D^{90°}$ and $R_E^{270°}$. Get a carry $T_{\overrightarrow{AC}}$ in which the point D goes into F, and $\overrightarrow{DF} = \overrightarrow{AC}$. But $\angle FDE = 45°$, and, therefore, the desired angle is also equal to 45°. **378.** Let $T_{\overrightarrow{CD}}(B) = B'$, and E be the midpoint of the line segment AB'. Through the points A, B, C, and D draw straight lines parallel to BE. **379.** Let $BC \parallel AD$. Consider the image of the diagonal AC in the carry $T_{\overrightarrow{AD}}$. **380.** Let $BC \parallel AD$. Consider the images of the line segments AB and CD in the carries $T_{\overrightarrow{BM}}$ and $T_{\overrightarrow{CM}}$. Prove that the bisector of the angle of the obtained triangle is at the same time a median (Fig. 185). **381.** Consider the carry $T_{\overrightarrow{AC}}$. Constructing the corresponding point L for the point K, find that $AK \parallel CL$, $\angle KCL = 90°$, and, consequently, $\angle AKC = 90°$. **382.** Carrying the lateral side in the direction and over the distance determined by the base of the trapezoid, get a triangle. Compute the altitude of the triangle. **383.** Note that the sums of the distances of the opposite vertices of the parallelogram $OABC$ to any straight line are equal, since these sums are equal to twice the distance of the point of intersection of the diagonals to this straight line. **384.** Construct an auxiliary circle, which is equal to the given circle, touches it, and is passed through the point M (Fig. 186). Note that $OO_1 = 2R$ and $MO_1 = R$. **386.** Let AD and BC be the bases of the trapezoid $ABCD$, M the midpoint of the line segment BC, and N the midpoint of the line segment AD. In the translation $T_{\overrightarrow{BM}}$ the point B goes into the point M, and the point A into the point A_1. In the translation $T_{\overrightarrow{CM}}$ the point C goes into the point M, and the point D into the point D_1. Then $A_1N = AN - AA_1 = AN - BM$ (1), $ND_1 = ND - D_1D = ND - MC$ (2). Adding together Equalities (1) and (2), find that $A_1N + ND_1 = AN + ND - (BM + MC) = AD - BC$. But

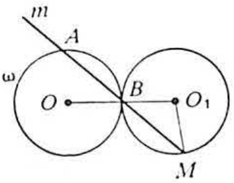


Fig. 186

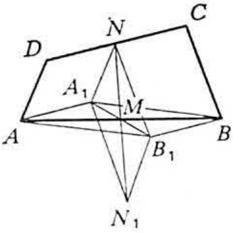


Fig. 187

$A_1N + ND_1 = A_1D_1$, and, hence, $A_1D_1 = AD - BC$ (3). Since $A_1N = ND_1$, MN is a median of the formed right triangle A_1MD_1. Therefore, $MN = \frac{1}{2}A_1N + ND_1$. Taking into account Equality (3), find that $MN = \frac{1}{2}A_1D_1 = \frac{1}{2}(AD - BC)$. 387. Denote the vertices of the trapezoid by A, B, C, and D ($AC = 13$ cm, $BD = 20$ cm, and $AD + BC = 21$ cm). Then $S_{ABCD} = \frac{1}{2}(AD + BC)\,h$, where h is the altitude of the trapezoid. Consider the translation $T_{\overrightarrow{BC}}$. This translation carries the point B into the point C, and the point D into the point D'. The area of the triangle ACD' is equal to $\frac{1}{2}AD'h$. But $AD' = AD + DD' = AD + BC$, and, therefore, the area of the trapezoid $ABCD$ is also equal to the area of the triangle ACD'. 388. Denote the centres of the given circles by O_1 and O_2. Then the translation $T_{\overrightarrow{O_1O_2}}$ carries the circle with centre O_1 into the circle with centre O_2. In this translation the point A goes into the point C, and the point B into the point D. Consequently, $AC = BD = O_1O_2 = d$. 389. Suppose the desired quadrilateral $ABCD$ is constructed (Fig. 187). Carry out the translation $T_{\overrightarrow{DN}}$ of the side DA and the translation $T_{\overrightarrow{CN}}$ of the side CB. Now, emanating from the point N are three line segments NA_1, NM, and NB_1 of known length. It is easy to show that M is the midpoint of the line segment A_1B_1. Indeed, the lengths of the line segments AA_1 and BB_1 are equal to $\frac{1}{2}DC$, these line segments being parallel to DC. Therefore, the quadrilateral A_1AB_1B is a parallelogram. The point M is the midpoint of its diagonal AB. Therefore, M belongs to the diagonal A_1B_1 and is its midpoint. Thus, in the triangle NA_1B_1 the sides NA_1, NB_1, and the median enclosed between them are known. To construct this triangle, mark the point N_1 symmetric with respect to N, M being the centre of symmetry. Obviously, $A_1N_1 = NB_1$. The triangle NN_1A_1 can be constructed by three known sides, that is, $NA_1 = DA$, $A_1N_1 = NB_1 = CB$, and $NN_1 = 2NM$. Now construct the required quadrilateral. Divide the line segment NN_1 by the point M into two equal parts. Construct the point B_1 symmetric with respect to the point A_1, M being the centre of symmetry. Construct the triangles A_1MA and B_1MB by three sides. Carrying the line segment AA_1 by $T_{\overrightarrow{A_1N}}$, and the line segment BB_1 by $T_{\overrightarrow{B_1N}}$, get all the four vertices of the desired quadrilateral $ABCD$. It is not difficult to show the uniqueness of the solution. 391. Consider the general case: the triangle ABC is irregular, and, hence, the points O, H, and M are distinct. The homothetic transformation with centre M and the ratio of similitude

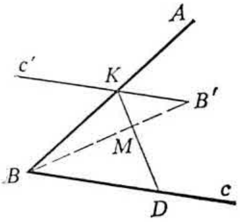


Fig. 188

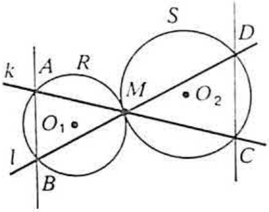


Fig. 189

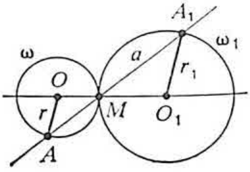


Fig. 190

$k=-\frac{1}{2}$ carries the triangle ABC into the triangle $A'B'C'$ whose vertices are the midpoints of the sides of the given triangle. The corresponding sides of these triangles are parallel. The altitudes AA_1, BB_1, and CC_1 of $\triangle ABC$ go into the altitudes $A'A_1'$, $B'B_1'$, and $C'C_1'$ of $\triangle A'B'C'$, the altitudes of the triangle $A'B'C'$ being perpendicular to the sides of the triangle ABC drawn through their midpoints. Hence, in the indicated homothetic transformation, the point H of intersection of the altitudes of the triangle goes into the centre O of the circle circumscribed about the triangle ABC. Hence it follows that the points M (the centre of the homothetic transformation), H, and O (the corresponding points in the homothetic transformation) lie in one straight line, and $\overrightarrow{MO}=-\frac{1}{2}\overrightarrow{MH}$, whence $\overrightarrow{OM}=\frac{1}{2}\overrightarrow{MH}$. If ABC is a regular triangle, then $O=M=H$, and Euler's line is indeterminate. 392. Let KD be the desired line segment, that is, $KM:MD=1:2$ (Fig. 188). Then the homothetic transformation $H_M^{-\frac{1}{2}}$ carries the point D into the point K. Since D belongs to BC, K belongs to $B'C'$, where $B'C'=H_M^{-\frac{1}{2}}(BC)$. Consequently, K is the point of intersection of BA and $B'C'$. On constructing the point K, find on BC the point D, which is the preimage of the point K in the homothetic transformation $H_M^{-\frac{1}{2}}$. 393. Two circles O_1 and O_2 touching each other at the point M are homothetic with respect to this point. Consider the homothetic transformation which carries R into S. This transformation carries the point A into the point C (Fig. 189), and the point B into the point D. Using the properties of homothety, find that $AB \parallel CD$. 394. Let M be the point of contact of the circle ω with centre O and radius r and the circle ω_1 with centre O_1 and radius r_1, and a the secant intersecting the circles for the second time at the points A and A_1 (Fig. 190). It is required to prove that $O_1A_1 \parallel OA$. Consider the homothetic transformation H_M carrying the point O into the point O_1. In this homothetic transformation, the straight line a goes into itself since it is passed through the centre, and the circle ω into the circle ω_1. The point A belonging to the intersection of a and ω goes into the point belonging to the intersection of a and ω_1, but different from M, and, hence, into the point A_1. Since O_1A_1 is the image of the line segment OA in a homothetic transformation, these line segments are parallel. 395. Denote the point of intersection of the straight lines MP and NQ by X, the point of intersection of the straight lines MR and NS by Y, and the point of intersection of the straight lines PR and QS by Z. It is easy to note that the composition of the homothetic transformations $H_X^{(M,P)}$ and $H_Z^{(P,R)}$ is a homothetic transformation $H_Y^{(M,R)}$, $H_X^{(M,P)}$ denoting the homothetic transformation with centre X carrying the point M into the point

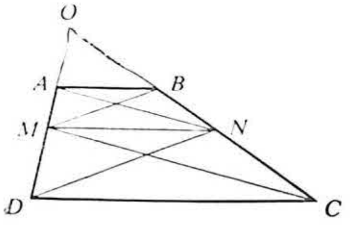


Fig. 191

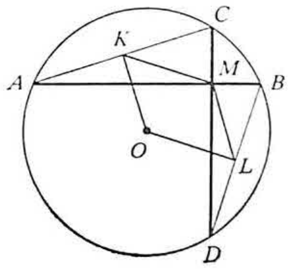


Fig. 192

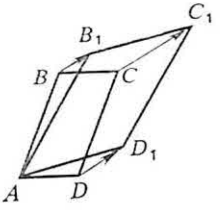


Fig. 193

P. But if the composition of two homothetic transformations is a homothetic transformation, then the centres of all the three homothetic transformations belong to one straight line, that is, Y belongs to ZX. The second part of the problem is solved in a similar way. **396.** Consider the homothetic transformation H_A carrying the circle ω into the circle ω_1. In this case, the points M and N go into the respective points N_1 and P_1 (N_1 is the point of intersection of the circle ω and the straight line AN, and P_1 is the point of intersection of the circle ω and the straight line AP). Then the straight line N_1P_1, as the image of the straight line NP, is parallel to it in the homothetic transformation. By virtue of the parallelism of the straight lines MQ and M_1P_1, the arcs MN_1 and QP_1 are equal, and, hence, the corresponding inscribed angles MAN_1 and QAP_1 are also equal, that is, $\angle MAN = \angle QAP$. **397.** Let M be a common end point of the line segments, $A_1, A_2, A_3, \ldots$ points of the straight line, which are other end points of these line segments, and M_1, M_2, and M_3 points dividing the line segments MA_1, MA_2, MA_3, ... in the given ratio λ, i.e. $\frac{A_1M_1}{M_1M}=\frac{A_2M_2}{M_2M}=\frac{A_3M_3}{M_3M}=\cdots=\lambda$. Show that $\frac{MM_1}{MA_1}=\frac{MM_2}{MA_2}=\frac{MM_3}{MA_3}=\cdots=$ $\frac{MA_1}{MM_1}=\frac{MM_1+M_1A_1}{MM_1}=1+\frac{M_1A_1}{MM_1}=1+\lambda$, then $\frac{MM_1}{MA_1}=\frac{1}{1+\lambda}$. Analogously, $\frac{MM_2}{MA_2}=\frac{1}{1+\lambda}$, etc. Consider $H_M^{\frac{1}{1+\lambda}}$, $H_M^{\frac{1}{1+\lambda}}(A_1)=M_1$, $H_M^{\frac{1}{1+\lambda}}(A_2)=M_2$, etc. Taking into consideration that the image of a straight line in a homothetic transformation is a straight line, find that the points M_1, M_2, M_3, etc. belong to one straight line. **413.** Let DA intersect CB at the point O Fig. 191). Then $\overrightarrow{OD}=\alpha\overrightarrow{OA}$, and $\overrightarrow{OC}=\alpha\overrightarrow{OB}$. By hypothesis, $AN \parallel CM$, and, hence, $\overrightarrow{OM}=\beta\overrightarrow{OA}$ and $\overrightarrow{ON}=\beta\overrightarrow{OC}$. Consequently, $\overrightarrow{OD}=\frac{\alpha}{\beta}\overrightarrow{OM}$, and $\overrightarrow{ON}=\frac{\alpha}{\beta}\overrightarrow{OB}$. Hence, $DN \parallel MB$. **415.** K is the midpoint of the chord AC, therefore, $\overrightarrow{OK}=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OC})$ (Fig. 192), and L is the midpoint of the chord BD, hence, $\overrightarrow{OL}=\frac{1}{2}(\overrightarrow{OB}+\overrightarrow{OD})$, and $\overrightarrow{OM}=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD})$. Hence, $\overrightarrow{LM}=\overrightarrow{OM}-\overrightarrow{OL}=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OC})$. Thus, $\overrightarrow{OK}=\overrightarrow{LM}$, and, hence, the quadrilateral $OKML$ is a parallelogram. **416.** Let CC_1 be a median of the triangle ABC, then $\overrightarrow{CC_1}=$

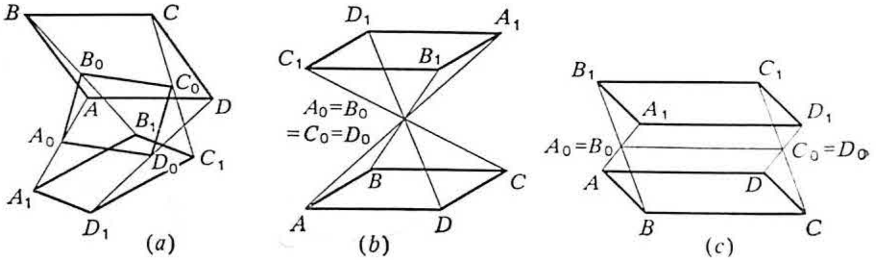


Fig. 194

$\frac{1}{2}(\overrightarrow{CA}+\overrightarrow{CB})$ and $|\overrightarrow{CC_1}|=\frac{1}{2}|\overrightarrow{CA}+\overrightarrow{CB}|$. Since $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are noncollinear, $|\overrightarrow{CA}+\overrightarrow{CB}|<|\overrightarrow{CA}|+|\overrightarrow{CB}|$. Hence, $CC_1<\frac{1}{2}(CA+CB)$. **417.** It is known that $\overrightarrow{MA}+\overrightarrow{MB}+\overrightarrow{MC}=0$, and, hence, $\overrightarrow{OM}=\frac{1}{3}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})$, O being an arbitrary point of the plane. Since $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are noncollinear vectors, $|\overrightarrow{OM}|<\frac{1}{3}(|\overrightarrow{OA}|+|\overrightarrow{OB}|+|\overrightarrow{OC}|)$. Hence, $OM<\frac{1}{3}(OA+OB+OC)$. **418.** By hypothesis, $ABCD$ and $AB_1C_1D_1$ are parallelograms, therefore, $\overrightarrow{AC_1}=\overrightarrow{AB_1}+\overrightarrow{AD_1}$ and $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{AD}$ (Fig. 193). Subtract the second vector equality from the first: $\overrightarrow{AC_1}-\overrightarrow{AC}=\overrightarrow{AB_1}-\overrightarrow{AB}+\overrightarrow{AD_1}-\overrightarrow{AD}$. Consequently, $\overrightarrow{CC_1}=\overrightarrow{BB_1}+\overrightarrow{DD_1}$, whence $CC_1\leqslant BB_1+DD_1$. **419.** Let A_0, B_0, C_0, and D_0 be the midpoints of the line segments AA_1, BB_1, CC_1, and DD_1, respectively (Fig. 194*a*). Then for an arbitrary point O in the plane $\overrightarrow{OA_0}=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OA_1})$, $\overrightarrow{OB_0}=\frac{1}{2}(\overrightarrow{OB}+\overrightarrow{OB_1})$, $\overrightarrow{OC_0}=\frac{1}{2}(\overrightarrow{OC}+\overrightarrow{OC_1})$, and $\overrightarrow{OD_0}=\frac{1}{2}(\overrightarrow{OD}+\overrightarrow{OD_1})$. Hence, $\overrightarrow{A_0B_0}=\overrightarrow{OB_0}-\overrightarrow{OA_0}=\frac{1}{2}(\overrightarrow{OB}-\overrightarrow{OA}+\overrightarrow{OB_1}-\overrightarrow{OA_1})=\frac{1}{2}(\overrightarrow{AB}+\overrightarrow{A_1B_1})$, $\overrightarrow{D_0C_0}=\overrightarrow{OC_0}-\overrightarrow{OD_0}=\frac{1}{2}(\overrightarrow{OC}-\overrightarrow{OD}+\overrightarrow{OC_1}-\overrightarrow{OD_1})=\frac{1}{2}(\overrightarrow{DC}+\overrightarrow{D_1C_1})$. By hypothesis, $ABCD$ and $A_1B_1C_1D_1$ are parallelograms, therefore, $\overrightarrow{AB}=\overrightarrow{DC}$ and $\overrightarrow{A_1B_1}=\overrightarrow{D_1C_1}$, consequently, $\overrightarrow{A_0B_0}=\overrightarrow{D_0C_0}$, and the quadrilateral $A_0B_0C_0D_0$ is a parallelogram. The required particular cases of the mutual arrangement of the given parallelograms are shown in Fig. 194*b* and *c*. **420.** Let $\frac{AP}{PB}=\frac{BQ}{QC}=\frac{CR}{RD}=\frac{DS}{SA}=k$, and O be an arbitrary point in the plane. $\frac{\overrightarrow{AP}}{\overrightarrow{PB}}=k$, whence $\frac{\overrightarrow{OP}-\overrightarrow{OA}}{\overrightarrow{OB}-\overrightarrow{OP}}=k$, and, hence, $\overrightarrow{OP}=\frac{\overrightarrow{OA}+k\overrightarrow{OB}}{k+1}$. Analogously, $\overrightarrow{OQ}=\frac{\overrightarrow{OB}+k\overrightarrow{OC}}{k+1}$, $\overrightarrow{OR}=\frac{\overrightarrow{OC}+k\overrightarrow{OD}}{k+1}$,

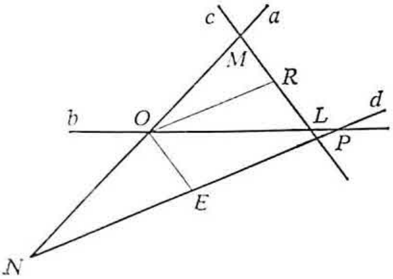


Fig. 195

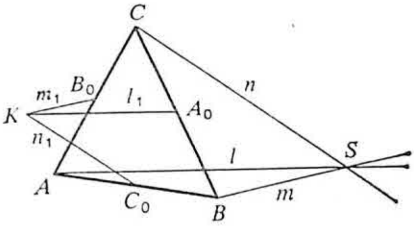


Fig. 196

and $\overrightarrow{OS}=\dfrac{\overrightarrow{OD}+k\overrightarrow{OA}}{k+1}$. It is easy to note that $\overrightarrow{PQ}=\overrightarrow{SR}$. Indeed, $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\dfrac{1}{k+1}(\overrightarrow{OB}-\overrightarrow{OA})+\dfrac{k}{k+1}(\overrightarrow{OC}-\overrightarrow{OB})=\dfrac{1}{k+1}\overrightarrow{AB}+\dfrac{k}{k+1}\overrightarrow{BC}$ and $\overrightarrow{SR}=\overrightarrow{OR}-\overrightarrow{OS}=\dfrac{1}{k+1}(\overrightarrow{OC}-\overrightarrow{OD})+\dfrac{k}{k+1}(\overrightarrow{OD}-\overrightarrow{OA})=\dfrac{1}{k+1}\overrightarrow{DC}+\dfrac{k}{k+1}\overrightarrow{AD}$. But $\overrightarrow{AB}=\overrightarrow{DC}$ and $\overrightarrow{BC}=\overrightarrow{AD}$, therefore, $\overrightarrow{PQ}=\overrightarrow{SR}$, and, consequently, $PQRS$ is a parallelogram. **421.** Denote the points of intersection of the medians of the triangles ABC and $A_1B_1C_1$ by G and G_1. Let $AG \| B_1C_1$, $BG \| A_1C_1$, and $CG \| A_1B_1$. Then, by hypothesis, $\overrightarrow{GA}=\lambda(\overrightarrow{G_1B_1}-\overrightarrow{G_1C_1})$, $\overrightarrow{GB}=\mu(\overrightarrow{G_1C_1}-\overrightarrow{G_1A_1})$, and $\overrightarrow{GC}=\gamma(\overrightarrow{G_1A_1}-\overrightarrow{G_1B_1})$. Adding these equalities termwise, we get: $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=\overrightarrow{G_1A_1}(\gamma-\mu)+\overrightarrow{G_1B_1}(\lambda-\gamma)+\overrightarrow{G_1C_1}(\mu-\lambda)$. But $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=\mathbf{0}$, and, therefore, $\overrightarrow{G_1A_1}(\gamma-\mu)+\overrightarrow{G_1B_1}(\lambda-\gamma)+\overrightarrow{G_1C_1}(\mu-\lambda)=0$. On the other hand, $\overrightarrow{G_1A_1}+\overrightarrow{G_1B_1}+\overrightarrow{G_1C_1}=\mathbf{0}$. Hence, find that $\lambda=\gamma=\mu$. Thus, $\dfrac{1}{\lambda}(\overrightarrow{GA}-\overrightarrow{GB})=\overrightarrow{G_1A_1}+\overrightarrow{G_1B_1}-2\overrightarrow{G_1C_1}=3(\overrightarrow{G_1A_1}+\overrightarrow{G_1B_1})=-3\overrightarrow{G_1C_1}$, i.e. $\dfrac{1}{\lambda}\overrightarrow{BA}=-3\overrightarrow{G_1C_1}$, whence it follows that $BA \| G_1C_1$. **422.** Let A_1, B_1, C_1, and D_1 be the respective points of intersection of the medians of the triangles BCD, CDA, DAB, and ABC; $\overrightarrow{OA_1}=\dfrac{1}{3}(\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD})$, $\overrightarrow{OB_1}=\dfrac{1}{3}(\overrightarrow{OC}+\overrightarrow{OD}+\overrightarrow{OA})$, $\overrightarrow{OC_1}=\dfrac{1}{3}(\overrightarrow{OD}+\overrightarrow{OA}+\overrightarrow{OB})$, and $\overrightarrow{OD_1}=\dfrac{1}{3}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})$ (1). If M and M_1 are the respective points of intersection of the midlines of the quadrilaterals $ABCD$ and $A_1B_1C_1D_1$, then $\overrightarrow{OM}=\dfrac{1}{4}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD})$. Taking into account Equality (1), we get: $\overrightarrow{OM_1}=\dfrac{1}{4}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD})$, which proves that $M_1=M$. **423.** Let R be the midpoint of ML and it is given that $d \| OR$ (Fig. 195). Prove, for instance, that in this case $c \| OE$, where E is the midpoint of NP. Then, $\overrightarrow{OP}=m\overrightarrow{OL}$, $\overrightarrow{ON}=n\overrightarrow{OM}$, and $NP \| OR$, i.e. $\overrightarrow{OP}-\overrightarrow{ON}=k(\overrightarrow{OM}+\overrightarrow{OL})$. Substituting the values of $\overrightarrow{OP}$ and $\overrightarrow{ON}$ into the last equality, obtain: $m\overrightarrow{OL}-$

$n\overrightarrow{OM}=k\overrightarrow{OM}+k\overrightarrow{OL}$, i.e. $(m-k)\overrightarrow{OL}=(k+n)\overrightarrow{OM}$. Since the vectors $\overrightarrow{OL}$ and $\overrightarrow{OM}$ are noncollinear, $m-k=0$ and $k+n=0$, i.e. $m=k$ and $n=-k$. Then $\overrightarrow{OP}=k\overrightarrow{OL}$ and $\overrightarrow{ON}=-k\overrightarrow{OM}$. Consider the vector $(\overrightarrow{ON}+\overrightarrow{OP})$, collinear with $\overrightarrow{OE}$. $\overrightarrow{ON}+\overrightarrow{OP}=-k\overrightarrow{OM}+k\overrightarrow{OL}=k(\overrightarrow{OL}-\overrightarrow{OM})=k\overrightarrow{ML}$. Consequently, $OE\|ML$. Similarly, it is also possible to prove the indicated property for other straight lines. **424.** Let the straight lines m_1 and l_1 intersect at the point K (Fig. 196). Prove that $\overrightarrow{KC_0}\|n$. $\overrightarrow{KC_0}=\overrightarrow{SC_0}-\overrightarrow{SK}$, $\overrightarrow{SC_0}=\frac{1}{2}\overrightarrow{SA}+\frac{1}{2}\overrightarrow{SB}$, and $\overrightarrow{SK}=\overrightarrow{SA_0}+\overrightarrow{A_0B_0}+\overrightarrow{B_0K}=\frac{1}{2}\overrightarrow{SB}+\frac{1}{2}\overrightarrow{SC}+\frac{1}{2}\overrightarrow{SA}-\frac{1}{2}\overrightarrow{SB}+m\overrightarrow{SB}=m\overrightarrow{SB}+\frac{1}{2}\overrightarrow{SA}+\frac{1}{2}\overrightarrow{SC}$. On the other hand, $\overrightarrow{SK}=\overrightarrow{SB_0}+\overrightarrow{B_0A_0}+\overrightarrow{A_0K}=\frac{1}{2}\overrightarrow{SB}+\frac{1}{2}\overrightarrow{SC}+\frac{1}{2}\overrightarrow{SA}-\frac{1}{2}\overrightarrow{SA}+n\overrightarrow{SA}=n\overrightarrow{SA}+\frac{1}{2}\overrightarrow{SB}+\frac{1}{2}\overrightarrow{SC}$. Then $m\overrightarrow{SB}+\frac{1}{2}\overrightarrow{SA}+\frac{1}{2}\overrightarrow{SC}=n\overrightarrow{SA}+\frac{1}{2}\overrightarrow{SB}+\frac{1}{2}\overrightarrow{SC}$, or $m\overrightarrow{SB}+\frac{1}{2}\overrightarrow{SA}=n\overrightarrow{SA}+\frac{1}{2}\overrightarrow{SB}$, whence $m=\frac{1}{2}$ and $n=\frac{1}{2}$. Therefore, $\overrightarrow{SK}=\frac{1}{2}(\overrightarrow{SA}+\overrightarrow{SB}+\overrightarrow{SC})$. Then $\overrightarrow{KC_0}=\frac{1}{2}\overrightarrow{SA}+\frac{1}{2}\overrightarrow{SB}-\frac{1}{2}(\overrightarrow{SA}+\overrightarrow{SB}+\overrightarrow{SC})=-\frac{1}{2}\overrightarrow{SC}$, and, hence, $KC_0\|SC$, that is, the straight line n_1 is passed through the point of intersection of the straight lines l_1 and m_1. **425.** See the solution of Problem 424. **426.** First prove that in any quadrilateral $\overrightarrow{MN}=\frac{1}{2}(\overrightarrow{CD}+\overrightarrow{BA})$, where M and N are the respective midpoints of CB and DA, $\overrightarrow{MN}=\overrightarrow{MC}+\overrightarrow{CD}+\overrightarrow{DN}$ and $\overrightarrow{MN}=\overrightarrow{MB}+\overrightarrow{BA}+\overrightarrow{AN}$. Adding these equalities and taking into account that $\overrightarrow{MC}+\overrightarrow{MB}=\mathbf{0}$, $\overrightarrow{DN}+\overrightarrow{AN}=\mathbf{0}$, get: $\overrightarrow{MN}=\frac{1}{2}(\overrightarrow{CD}+\overrightarrow{BA})$. If the vectors $\overrightarrow{CD}$ and $\overrightarrow{BA}$ are noncollinear, then $MN<\frac{1}{2}(CD+BA)$, which is readily proved, using the property of the sides of a triangle. Consequently, $CD\|BA$, that is, the given quadrilateral $ABCD$ is a trapezoid or a parallelogram. **427.** $\overrightarrow{AA_1}+\overrightarrow{BB_1}+\overrightarrow{CC_1}=0$. Therefore, $\overrightarrow{OA_1}-\overrightarrow{OA}+\overrightarrow{OB_1}-\overrightarrow{OB}+\overrightarrow{OC_1}-\overrightarrow{OC}=\mathbf{0}$. If O is the point of intersection of the medians of the triangle ABC, then $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\mathbf{0}$, and, consequently, $\overrightarrow{OA_1}+\overrightarrow{OB_1}+\overrightarrow{OC_1}=\mathbf{0}$. Hence it follows that O is the point of intersection of the medians of the triangle $A_1B_1C_1$. **428.** $\overrightarrow{CA_2}=\overrightarrow{A_2A}+\overrightarrow{AC}$. When rotated through an angle of 90°, $\overrightarrow{A_2A}$ goes into $\overrightarrow{CB}$ and $\overrightarrow{AC}$ into $\overrightarrow{BB_2}$, that is, $\overrightarrow{CA_2}$ goes into $\overrightarrow{CB}+\overrightarrow{BB_2}=\overrightarrow{CB_2}$, whence it follows that $CA_2\perp CB_2$ and $CA_2=CB_2$. **429.** Let D be the midpoint of C_1C_1' (Fig. 197), and $\overrightarrow{CD}=\frac{1}{2}(\overrightarrow{CC_1}+\overrightarrow{CC_1'})$. When rotated through an angle of 90°, the vectors $\overrightarrow{CC_1}$ and $\overrightarrow{CC_1'}$ go into the

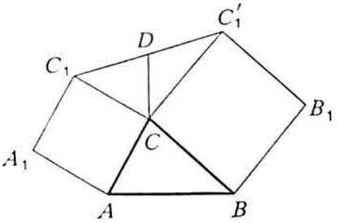


Fig. 197

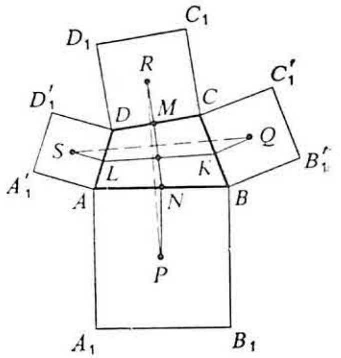


Fig. 198

vectors $\overrightarrow{CA}$ and $-\overrightarrow{CB}$, respectively. Consequently, this rotation carries the sum of the vectors $\overrightarrow{CC_1}+\overrightarrow{CC_1'}$ into the vector $\overrightarrow{CA}-\overrightarrow{CB}$, that is, into the vector $\overrightarrow{BA}$. Therefore, $(\overrightarrow{CC_1}+\overrightarrow{CC_1'}) \perp \overrightarrow{BA}$ and $|\overrightarrow{CC_1}+\overrightarrow{CC_1'}| = |\overrightarrow{BA}|$, whence the statement of the problem follows. **430.** Prove that the vector $\overrightarrow{PR}$ is obtained by rotating the vector $\overrightarrow{SQ}$ through an angle of 90° (Fig. 198). $\overrightarrow{SQ}=\overrightarrow{SL}+\overrightarrow{LK}+\overrightarrow{KQ}$, or $\overrightarrow{SQ}=\overrightarrow{SL}+\frac{1}{2}(\overrightarrow{DC}+\overrightarrow{AB})+\overrightarrow{KQ}$. Rotate each vector of the right-hand side of the last equality through an angle of 90°. Then $\overrightarrow{SL}$ goes into $\overrightarrow{LD}=\frac{1}{2}\overrightarrow{AD}$, $\frac{1}{2}\overrightarrow{DC}$ into $\overrightarrow{MR}$, $\frac{1}{2}\overrightarrow{AB}$ into $\overrightarrow{PN}$, and $\overrightarrow{KQ}$ into $\frac{1}{2}\overrightarrow{BC}$. Then $\overrightarrow{SQ}$ goes into $\frac{1}{2}\overrightarrow{AD}+\overrightarrow{MR}+\overrightarrow{PN}+\frac{1}{2}\overrightarrow{BC}=\overrightarrow{PN}+\frac{1}{2}(\overrightarrow{AD}+\overrightarrow{BC})+\overrightarrow{MR}=\overrightarrow{PN}+\overrightarrow{NM}+\overrightarrow{MR}=\overrightarrow{PR}$. Hence, $\overrightarrow{PR}$ is obtained by rotating $\overrightarrow{SQ}$ through 90°. Consequently, $PR \perp SQ$ and $PR=SQ$. **431.** Since $\frac{\overrightarrow{MA}+\overrightarrow{MB}}{2}=-\frac{\overrightarrow{MC}+\overrightarrow{MD}}{2}$, $\overrightarrow{MA}+\overrightarrow{MB}+\overrightarrow{MC}+\overrightarrow{MD}=\mathbf{0}$, i.e. $\overrightarrow{MD}=-(\overrightarrow{MA}+\overrightarrow{MB}+\overrightarrow{MC})$. Then $\vec{U}=\overrightarrow{MA}+\overrightarrow{MB}$ and $\vec{V}=\overrightarrow{MA}+\overrightarrow{MB}+\overrightarrow{MC}=-\overrightarrow{MD}$, i.e. $\vec{V}=-\overrightarrow{MD}$. This means that M is the midpoint of the line segment DV. Express the area of the quadrilateral $MAUV$ in the following ways: (a) $S_{MAUV}=S_{MAV}+S_{MUV}=S_{MAB}+S_{MCD}$; (b) $S_{MAUV}=S_{MAV}+S_{UAV}=S_{MAD}+S_{MBC}$. Hence, $2S_{MAUV}=S_{MAB}+S_{MCD}+S_{MAD}+S_{MBC}=S_{ABCD}$. Thus, $S_{ABCD}:S_{MAUV}=2$. **432.** Let AL and BK be medians of the triangle ABC, and M the point of intersection of AL and BK, then $\overrightarrow{KL}=\frac{1}{2}\overrightarrow{AB}$ (1). Further, $\overrightarrow{KL}=\overrightarrow{KM}+\overrightarrow{ML}$ (2). From Equalities (1) and (2) it follows that $\overrightarrow{KM}+\overrightarrow{ML}=\frac{1}{2}\overrightarrow{AB}$, hence, $2\overrightarrow{KM}+2\overrightarrow{ML}=\overrightarrow{AB}$, but $\overrightarrow{AM}+\overrightarrow{MB}=\overrightarrow{AB}$, and, therefore, $2\overrightarrow{KM}+2\overrightarrow{ML}=\overrightarrow{AM}+\overrightarrow{MB}$, or $(\overrightarrow{AM}-2\overrightarrow{ML})+(\overrightarrow{MB}-2\overrightarrow{KM})=\mathbf{0}$. The vectors $\overrightarrow{AM}-2\overrightarrow{ML}$ and

$\overrightarrow{MB}-2\overrightarrow{KM}$ are respectively collinear with the vectors $\overrightarrow{AL}$ and $\overrightarrow{BK}$, therefore, $\overrightarrow{AM}-2\overrightarrow{ML}=0$ and $\overrightarrow{MB}-2\overrightarrow{KM}=0$, i.e. $\overrightarrow{AM}=2\overrightarrow{ML}$ and $\overrightarrow{MB}=2\overrightarrow{KM}$, hence, $\dfrac{|\overrightarrow{AM}|}{|\overrightarrow{MB}|}=2$ and $\dfrac{|\overrightarrow{ML}|}{|\overrightarrow{KM}|}=2$. 433. First of all, prove that the points M, N, and B belong to one and the same straight line. To this end, prove that the vectors $\overrightarrow{MN}$ and $\overrightarrow{NB}$ are collinear. $\overrightarrow{MN}=\overrightarrow{MA}+\overrightarrow{AN}$, $\overrightarrow{MA}=\frac{1}{5}\overrightarrow{DA}$, $\overrightarrow{AN}=\frac{1}{6}(\overrightarrow{AB}+\overrightarrow{AD})$, $\overrightarrow{MN}=\frac{1}{5}\overrightarrow{DA}+\frac{1}{6}\overrightarrow{AB}-\frac{1}{6}\overrightarrow{DA}=\frac{1}{6}\overrightarrow{AB}+\frac{1}{30}\overrightarrow{DA}=\frac{1}{30}(5\overrightarrow{AB}+\overrightarrow{DA})$, $\overrightarrow{NB}=\overrightarrow{NA}+\overrightarrow{AB}=\overrightarrow{AB}-\frac{1}{6}\overrightarrow{AB}-\frac{1}{6}\overrightarrow{AD}=\frac{5}{6}\overrightarrow{AB}+\frac{1}{6}\overrightarrow{DA}$, and $\overrightarrow{NB}=\frac{1}{6}(5\overrightarrow{AB}+\overrightarrow{DA})$. Hence, $\overrightarrow{NB}=5\overrightarrow{MN}$, and, consequently, the vectors $\overrightarrow{NB}$ and $\overrightarrow{MN}$ are collinear. The point N divides the line segment MB in the ratio 1:5. 434. In vector language, the problem requires to prove the fact that $\overrightarrow{MO}=\overrightarrow{ON}$, where O is the point of intersection of the line segments whose end points are the midpoints of the opposite sides of the quadrilateral $ABCD$, and M and N are the midpoints of the diagonals BD and AC. $\overrightarrow{MO}=\overrightarrow{MP}+\overrightarrow{PO}$, but $\overrightarrow{MP}=\frac{1}{2}\overrightarrow{DA}$, and, hence, $\overrightarrow{MO}=\frac{1}{2}\overrightarrow{DA}+\overrightarrow{FO}$ (1). Further, $\overrightarrow{ON}=\overrightarrow{OR}+\overrightarrow{RN}$, $\overrightarrow{RN}=\frac{1}{2}\overrightarrow{DA}$, and, consequently, $\overrightarrow{ON}=\frac{1}{2}\overrightarrow{DA}+\overrightarrow{OR}$ (2). Comparing Equalities (1) and (2) and bearing in mind that $\overrightarrow{PO}=\overrightarrow{OR}$, get: $\overrightarrow{MO}=\overrightarrow{ON}$. 435. If the line segments A_1A_2 and B_1B_2 lie on straight tines parallel to l_1, then in the case when $\overrightarrow{A_1A_2}=\overrightarrow{B_1B_2}$ there are infinitely many solutions, that is, it is sufficient to take on the straight line any segment C_1C_2 such that $\overrightarrow{C_1C_2}=\overrightarrow{A_1A_2}$. And if the given line segments are such that $\overrightarrow{A_1A_2}\neq\overrightarrow{B_1B_2}$, then the required points C_1 and C_2 do not exist obviously. Now suppose, for definiteness, that A_1A_2 is not parallel to l and that A_3 is the point of intersection of the straight lines A_1A_2 and l_1. On the straight line B_1B_2 construct two points M_1 and M_2 such that $B_1B_2{:}B_2M=A_2A_1{:}A_2A_3$, and denote by B_3 the one for which the following condition is fulfilled: if A_1 lies between two other points on the straight line A_1A_2, then B_1 between two other points on the straight line B_1B_2. Now, join the points B_3 and A_3 and, through the points B_1 and B_2, draw straight lines parallel to A_3B_3; their points C_1 and C_2 of intersection with the straight line l are just the required points. Indeed, $\dfrac{C_1C_2}{C_2A_3}=\dfrac{B_1B_2}{B_2B_3}=\dfrac{A_1A_2}{A_2A_3}$, whence, by means of a derived proportion, get: $\dfrac{C_1A_3}{C_2A_3}=\dfrac{A_1A_3}{A_2A_3}$. Then the angles $A_1C_1A_3$ and $A_2C_2A_3$ are congruent as the corresponding angles of similar triangles $A_1C_1A_3$ and $A_2C_2A_3$, so that $A_1C_1 \parallel A_2C_2$. 436. Consider the solution of this problem using the vector method (Fig. 199). By the property of the sum of the vectors represented by a closed polygonal

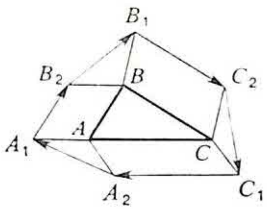


Fig. 199

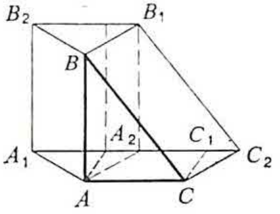


Fig. 200

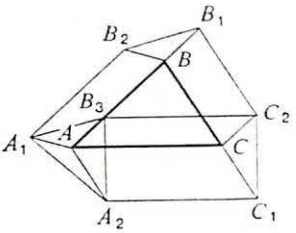


Fig. 201

line, obtain: $\overrightarrow{A_1B_2}+\overrightarrow{B_2B_1}+\overrightarrow{B_1C_2}+\overrightarrow{C_2C_1}+\overrightarrow{C_1A_2}+\overrightarrow{A_2A_1}=\mathbf{0}$. Using the associative and commutative properties of addition of vectors, get: $\overrightarrow{B_2B_1}+\overrightarrow{C_2C_1}+\overrightarrow{A_2A_1}+(\overrightarrow{A_1B_2}+\overrightarrow{B_1C_2}+\overrightarrow{C_1A_2})=\mathbf{0}$. Since $\overrightarrow{A_1B_2}=\overrightarrow{AB}$, $\overrightarrow{B_1C_2}=\overrightarrow{BC}$, $\overrightarrow{C_1A_2}=\overrightarrow{CA}$, and $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\mathbf{0}$, the sum of the vectors in brackets is also equal to a null vector. Therefore, $\overrightarrow{B_2B_1}+\overrightarrow{C_2C_1}+\overrightarrow{A_2A_1}=\mathbf{0}$, but then $\overrightarrow{B_2B_1}=\overrightarrow{C_1C_2}-\overrightarrow{A_2A_1}$. It is clear that any of the three given vectors can be represented as the difference of two others. If these vectors are noncollinear, then the required triangle exists. But two of the three given vectors cannot be collinear, since then the third vector will also be collinear with them (as the sum of two collinear vectors). But the case is possible when all the three vectors are collinear. Indeed, if the vertex A_1, e.g. of the third parallelogram, is represented by any point of the straight line parallel to C_1C_2, then get (Fig. 200): $\overrightarrow{B_2B_1}=\overrightarrow{B_2B}+\overrightarrow{BB_1}=\overrightarrow{A_1A}+\overrightarrow{CC_2}=(\overrightarrow{A_1A_2}+\overrightarrow{A_2A})+(\overrightarrow{CC_1}+\overrightarrow{C_1C_2})=(\overrightarrow{A_1A_2}+\overrightarrow{C_1C_2})+(\overrightarrow{A_2A}+\overrightarrow{CC_1})=\overrightarrow{A_1A_2}+\overrightarrow{C_1C_2}$. Thus, $\overrightarrow{B_2B_1}=\overrightarrow{A_1A_2}+\overrightarrow{C_1C_2}$, and the required triangle does not exist. Thus, the construction of this triangle is possible when the vectors corresponding to the given line segments are noncollinear. Consider the solution of the preceding problem (on the possibility of constructing a triangle) by the method of translation. Let $\overrightarrow{B_1B_2}$, $\overrightarrow{C_1C_2}$, and $\overrightarrow{A_1A_2}$ be noncollinear vectors. Carry out the translation $T_{\overrightarrow{CA}}$ of the triangle CC_1C_2 (Fig. 201). Then CC_2 goes into AB_3, CC_1 into AA_2, and C_1C_2 into A_2B_3. Indeed, $BB_1 \parallel CC_2$ and $BB_1=CC_2$, $AB_2 \parallel B_2B_1$ and $AB_2=B_2B_1$. By transitivity of the relations of parallelism and equality, obtain: $CC_2 \parallel AB_3$ and $CC_2=AB_3$. But then C_1C_2 goes into A_2B_3. On carrying out the translation $T_{\overrightarrow{BA}}$ of the triangle BB_1B_2, finally get: B_2 goes into A_1, B_1 into B_2, C_2 into B_3, and C_1 into A_2. Thus, the sides of the triangle $A_1B_3A_2$ are equal to the line segments B_2B_1, C_2C_1, and A_2A_1. **543.** (a) $2\sqrt{\dfrac{2S}{\sin\beta}}$; (b) $2\sqrt{S\tan\dfrac{\beta}{2}}$; (c) $2\left(1+\sin\dfrac{\beta}{2}\right)\sqrt{\dfrac{2S}{\sin\beta}}$. **544.** $\arccos\dfrac{4}{5}$. **545.** $b+\dfrac{c}{2}$. **546.** The least value is equal to h_c and the greatest to h_a. **547.** The diagonal is the altitude of the triangle. **548.** $4\sqrt{2}$ m. **549.** 30 cm. **550.** (a) 6000 cm^2; (b) 108 cm^2. **551.** $KD=MD=5$ cm and $S=32$ cm^2. **552.** (a) 45 cm^2; (b) 36.125 cm^2. **553.** (a), (b) $AC=BC$. **554.** (a), (b) The central angle of the sector is equal to 2. 555. $\frac{p}{\pi+4}$. 556. $\frac{\sqrt{h^2+8R^2}-3h}{4}$. 557. $\frac{3}{4}R^2\sqrt{3}$. 558. *First Method*. Let in the triangle ABC, $AC=b$ and $\angle B=\beta$. Denote $\angle BAC=x$, express AB and BC in terms of b, x, and β, and double the length of the median to construct a parallelogram. *Second Method* (geometrical). Circumscribe a circle about the triangle ABC and prove that the median BM of an arbitrary triangle is smaller (and in the case of an obtuse angle, larger) than the median B_1M of the isosceles triangle AB_1C. **559.** Let AP and CK be perpendicular to l. Denote $\angle PBA=x$. After completing computations, prove that $PB=BK$. **560.** (a) $\frac{2\sqrt{S}}{\sqrt[4]{27}}$; (b) $3\sqrt{3}\,r^2$. Denote half the angle at the base of the triangle by x. **561.** $\frac{75}{4}$ cm². Let $PKME$ be an inscribed trapezoid, the point E lying on BC, and the point M on CD. The triangles PAK and ECM are similar. Taking advantage of this fact, introduce the notation: $CM=2x$ and $CE=3x$. **562.** 100°. See Example 4 from Sec. 7.

Chapter 2

631. 60°. **632.** 60°. **633.** 115° and 35°. **634.** 30°. **635.** $2\arccos\left(\frac{\cos\alpha}{\cos\beta}\right)$. **636.** $\arcsin\frac{4}{5}$. **637.** $\arcsin\frac{\sqrt{6}}{3}$. **638.** 90°. **639.** $\arcsin\frac{6\sqrt{34}}{85}$. **640.** 30° and $\arctan\frac{\sqrt{51}}{17}$. **641.** $\arcsin\frac{4ab\sqrt{39}}{13(a^2+b^2)}$. **642.** $\arctan\frac{\sqrt{15}}{15}$. **643.** $\arcsin\frac{\sqrt{2}}{6}$. **644.** $\arcsin\frac{\sqrt{6}}{3}$. **645.** $\arcsin\frac{4\sqrt{65}}{65}$. **646.** $\arcsin\frac{\sqrt{15}}{5}$ and $\arcsin\frac{\sqrt{15}}{15}$. **647.** $\arcsin\frac{\sqrt{3}}{9}$. **648.** $\arcsin\frac{2\sqrt{30}}{15}$. **649.** $\arcsin\frac{2\sqrt{30}}{15}$. **650.** $\arcsin\left(\frac{\sin\alpha}{\cos\beta}\right)$. **651.** $\frac{R}{2}(2\sqrt{3}-\sqrt{2})$. **652.** 60°. **653.** The first line segment forms angles of 120°, 90°, and 60° with the second, third, and fourth line segments, respectively, the second line segment angles of 120° and 90° with the third and fourth respectively, and the third line segment an angle of 120° with the fourth. To find the angle AB_1C_2 (Fig. 202), construct AD_1, D_1C_2, and B_1D_1. Since $AD_1 \parallel B_1C_2$, the points A, B_1, C_2, and D_1 lie in one and the same plane. Consequently, $\angle AB_1C_2=\angle AB_1D_1+\angle D_1B_1C_2$. In order to find the angle between the straight lines BA_1 and C_2D_3, take advantage of the parallelism of the straight lines BA_1 and C_2D_3. **654.** 45°. **657.** $\arccos\frac{1}{6}$. Let CM be a median of the triangle ABC (Fig. 203). In the plane CEM, through the

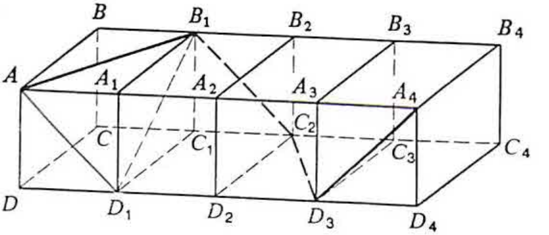


Fig. 202

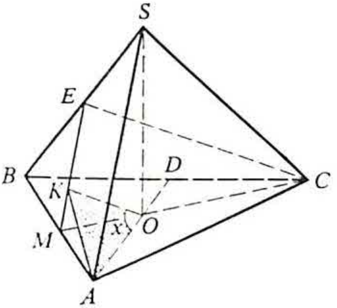


Fig. 203

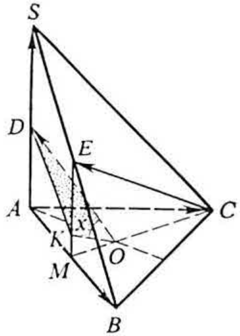


Fig. 204

point O draw the straight line $OK \parallel CE$. Then the angle AOK is equal to the angle between the straight lines AD and CE. The angle AOK can be found by computing the sides of the triangle AOK. It is advisable to take $AB=a$ as an auxiliary parameter. **658.** $\arccos \frac{5\sqrt{102}}{102}$. *First Method.* Let CM be a median of the triangle ABC (Fig. 204). In the plane CEM, through the point O draw the straight line $OK \parallel CE$. Then the angle KOD is equal to the sought-for angle between the straight lines OD and CE. The angle KOD can be found by computing the sides of the triangle KOD. *Second Method.* Since $AS \perp AC$, $AS \perp AB$, $AB \perp AC$, and $AB=AC=AS$, it is possible to introduce a rectangular Cartesian coordinate system, by setting $\overrightarrow{AC}=\mathbf{i}$, $\overrightarrow{AB}=\mathbf{j}$, and $\overrightarrow{AS}=\mathbf{k}$. Then $\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{AD}=\frac{2}{3}\overrightarrow{FA}+\overrightarrow{AD}=-\frac{1}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}+\frac{1}{2}\mathbf{k}$, and $\overrightarrow{CE}=\overrightarrow{CB}+\overrightarrow{BE}=-\frac{3}{2}\mathbf{i}+\mathbf{j}+\frac{1}{2}\mathbf{k}$. **659.** $\arccos(\sin\alpha\sin\beta+\cos\alpha\cos\beta\cos\gamma)$. **660.** 45° and 60°. **661.** $\arccos\frac{1}{5}$. Setting $\overrightarrow{DC}=\mathbf{i}$, $\overrightarrow{DA}=\mathbf{j}$, and $\overrightarrow{DD_1}=\mathbf{k}$, find that $\overrightarrow{DP}=\overrightarrow{DA}+\overrightarrow{AP}=\mathbf{j}+\frac{1}{2}\mathbf{k}$, $\overrightarrow{QC}=\overrightarrow{DC}+\frac{1}{2}\overrightarrow{CC_1}=\mathbf{i}+\frac{1}{2}\mathbf{k}$. **662.** $\arccos\frac{\sqrt{3}}{6}$. **663.** $\arccos\frac{\sqrt{10}}{8}$. **664.** $\arctan\sqrt{19}$. In the face SAB, through the point M draw the straight line $MF \parallel SK$ (Fig. 205). Then the desired angle is equal to $\angle DMF=x$. Setting $AB=a$ as an auxiliary parameter, find the sides FD and MF of the triangle DMF and then find $\tan x$. **665.** $\arctan\frac{\sqrt{3}}{2}$. **666.** $\arccos\frac{1}{4}$. **667.** $\arccos\left(-\frac{1}{4}\right)$. In the plane APQ, through the point D draw the straight line $DR \parallel AQ$ (Fig. 206). Then the angle BDR is equal to the angle between the rays DB and AQ. To determine this angle, find the sides of the triangle BDR. Taking $BC=a$ as an auxiliary parameter, find in the triangle BDR that $DR=a\sqrt{2}$ and $BR=a\sqrt{5}$. The side BD can be found from the right triangle BDM, where $BM=\frac{a\sqrt{3}}{2}$ and $DM=\frac{a\sqrt{5}}{2}$. Then $BD=a\sqrt{2}$, and, further, by the law of cosines,

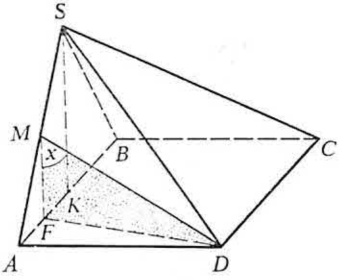


Fig. 205

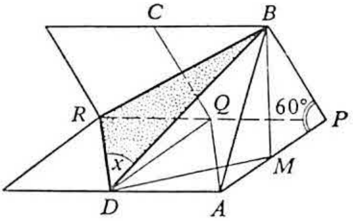


Fig. 206

find the angle BDR. 668. $\arccos\frac{\sqrt{10}}{10}$. Decompose the vectors $\overrightarrow{BP}$ and $\overrightarrow{OL}$ along the vectors $\mathbf{i}=\overrightarrow{OC}$, $\mathbf{j}=\overrightarrow{OD}$, and $\mathbf{k}=\overrightarrow{OS}$ of the rectangular Cartesian basis and then find the cosine of the angle between the vectors $\overrightarrow{BP}$ and $\overrightarrow{OL}$. Also proceed in a different way. In the plane SBD, through the point O draw the straight line $OM \parallel BP$. Then $\angle LOM$ is equal to the sought-for angle. To determine this angle, find the sides LM, OL, and OM of the triangle LOM and then apply the law of cosines. 669. $\arccos\left(-\frac{1}{10}\right)$. Decompose the vectors $\overrightarrow{KB}$ and $\overrightarrow{CD}$ along the vectors $\mathbf{i}=\overrightarrow{MA}$, $\mathbf{j}=\overrightarrow{MD}$, and $\mathbf{k}=\overrightarrow{MB}$ of the rectangular Cartesian basis, where the point M is the midpoint of the diagonal AC of the square $ABCD$. 670. $\arccos\frac{1}{4}$. 671. $\arccos\left(-\frac{\sqrt{6}}{8}\right)$. In the plane SBC, through the point S draw the straight line $l \parallel BD$, and through the point D the straight line $m \parallel SB$. Let the straight lines l and m intersect at the point K. Then $\angle KSO$ is equal to the desired angle. To determine the angle KSO, find the sides KO, SK, and SO of the triangle KSO and then apply the law of cosines. 672. $\arccos(\sin\alpha\sin\beta)$. 673. 4.8 cm. 674. $\frac{\sqrt{b^2-a^2}\sin\alpha\sin\beta}{|\sin(\alpha\pm\beta)|}$. 675. $\frac{a}{2}$. 676. $\frac{a\sqrt{21}}{7}$. 677. R and $\sqrt{R^2+h^2\sin^2\alpha}$. 678. $\frac{ab}{a+b}$. 679. $\frac{ah}{\sqrt{a^2+h^2}}$. 680. $a\sqrt{\frac{a^2\sin^2\varphi+h^2}{a^2+h^2}}$. 681. $\frac{a\sqrt{2}}{2}$. 682. $\frac{a\sqrt{3}}{3}$. 683. $\frac{R\sqrt{-\cos 2\alpha}}{\sin\alpha}$. 684. $\sqrt{b^2-a^2\cot^2\alpha}$. 685. $\sqrt{22.5}$ cm, $\sqrt{13.5}$ cm, and $\sqrt{2.5}$ cm. Cutting the pyramid along the edges SA, SB, and SC and then developing the lateral faces, as is shown in Fig. 207, get the triangle $S'S''S'''$. Indeed, since, when rotated, the size of the angle remains unchanged, $\angle 1 = \angle 3$, $\angle 2 = \angle 4$, and, consequently, $\angle 3 + \angle BAC + \angle 4 = 180°$, that is, the point A lies on the side $S'S''$. Analogously, the point B lies on the side $S'S''$, and the point C on the side $S''S'''$. In the triangle $S'S''S'''$, AB, AC, and BC are midlines, and, therefore, the triangles SAB, SBC, SAC, and ABC are the images of four congruent triangles. Then place the given pyramid in the parallelepiped, as is shown in Fig. 208, and, on having proved that this parallelepiped is

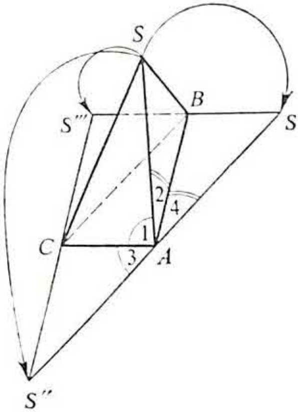


Fig. 207

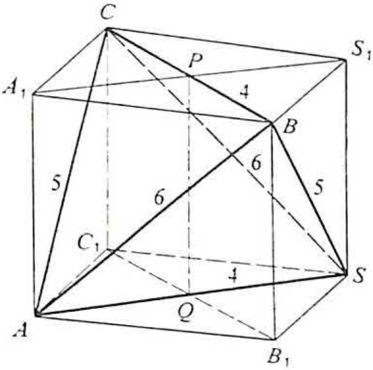


Fig. 208

rectangular, find the desired distances. **690.** 60°. **691.** 90°. **692.** 45°. **693.** $\arccos(\cot^2\alpha)$. **694.** $\arccos(\sin\beta\cos\alpha)$, where $0° < \alpha < 90°$. **695.** $\arcsin\left(\frac{2\sqrt{3}}{3}\sin\alpha\right)$. **696.** $\alpha_1 = \arccos\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}$, $\beta_1 = \arccos\frac{\cos\beta - \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma}$, and $\gamma_1 = \arccos\frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}$. **697.** $\arcsin\frac{\sqrt{a^2+b^2}}{2a}$. **698.** $\arcsin\left(\frac{\sin\alpha}{\sin\beta}\right)$. **699.** $\arcsin\left(\frac{\sqrt{2}}{2}\sin\alpha\right)$. **700.** $\arctan\frac{2\sqrt{6}}{3}$. From the point K drop the perpendicular KL to the plane ABC, and then from the point L the perpendicular LM to the median AD, and join the point M to the point K. The angle KML is the plane angle of the required dihedral angle. To determine the angle KML, find the legs KL and ML of the right triangle KML. **701.** $\arctan\frac{\sqrt{21}}{6}$. **702.** $\arccos\frac{17}{35}$. **703.** $2\arctan\left(\frac{1}{\sin\alpha}\right)$. **704.** 30°, 60°, 120°, and 60°. **705.** $\arctan\left(\frac{1}{2}\tan\alpha\right)$. **706.** $\arccos\frac{\sqrt{1-2\cos\alpha}}{3}$. **707.** $2\arcsin\frac{\sqrt{1+3\cos^2\alpha}}{2}$. **709.** $\arccos\sqrt{\frac{1-\cos\alpha}{3+\cos\alpha}}$. **710.** 45°. From the point L drop the perpendicular LN to the plane $ABCD$, and then from the point N the perpendicular NM to the straight line FK, and join the point L to the point N. The angle LMN is the plane angle of the desired dihedral angle. To determine the angle LMN, find the legs LN and MN of the right triangle LMN. **711.** $\arccos(-\cos^2\alpha)$. **712.** $\frac{180°}{n}$. **713.** $\arccos\left(\cot\frac{180°}{n}\cot\alpha\right)$. **714.** $\arcsin\left(\cot\frac{180°}{n}\cot\alpha\right)$. **715.** $\tan x = \frac{\cos\alpha}{\sqrt{-\cos\left(\frac{\pi}{n}+\alpha\right)\cos\left(\frac{\pi}{n}-\alpha\right)}}$. **716.** $\arccos\frac{1}{5}$. In the plane

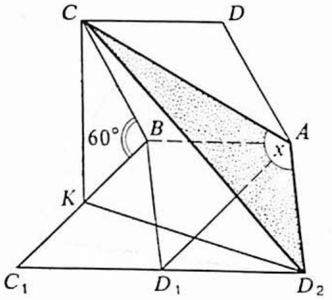


Fig. 209

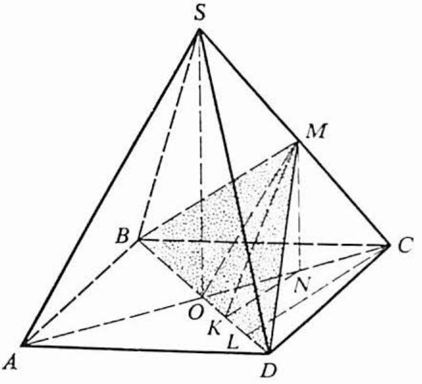


Fig. 210

ABC_1D_1, through the point A draw a straight line parallel to the straight line BD_1 and denote the point of its intersection with the straight line C_1D_1 by D_2 (Fig. 209). The angle CAD_2 is the desired angle. It can be determined in the following way. Setting, for instance, $AB=a$ as an auxiliary parameter, find the sides CD_2, AC, and AD_2 of the triangle CAD_2 and then apply the law of cosines to this triangle. **717.** $\arccos \frac{2+3\sqrt{2}}{8}$. **718.** 45°. **719.** $\arccos \frac{3\sqrt{2}-2}{8}$. **720.** $\arcsin \frac{3\sqrt{58}}{29}$. Drop the perpendicular CP from the point C to the plane SAB. Then the line segment DP is the projection of CP on the plane SAB, and, consequently, $\angle CDP$ is the desired angle. To determine the angle CDP, proceed in the following way. Setting $AB=a$ as an auxiliary parameter, find any two sides of the triangle CDP. **721.** $\arccos \frac{\sqrt{2}}{4}$ and $\arccos \frac{\sqrt{6}}{4}$. **722.** $\arccos \left(-\frac{\sqrt{6}}{6}\right)$. **723.** $2 \arctan \frac{\sqrt{7}}{3}$. **724.** a, $\frac{2}{3}a$, and $\frac{1}{3}a$. **725.** $\frac{a}{6}$. **726.** $\frac{ab\sqrt{2}}{4}$. **727.** $\frac{5a^2\sqrt{2}}{16}$. **728.** $3a^2$. **729.** $\frac{S}{a}$. **730.** $\frac{H^2 \sin 2\alpha}{8 \sin (30^\circ+\alpha) \sin (30^\circ-\alpha)}$. **731.** $\frac{3a\sqrt{17}}{17}$. **732.** $\frac{7a^2\sqrt{17}}{24}$. **733.** $\frac{7a^2\sqrt{113}}{72}$. **734.** $\frac{a^2\sqrt{19}}{10}$. **735.** $\frac{a^2\sqrt{51}}{8}$, $S_{\text{sec}}=\frac{1}{2} BD \cdot MK$, where $MK \perp BD$. The construction $MK \perp BD$ can be fulfilled in the following manner (Fig. 210): (1) in the plane SAC, through the point O draw $OM \parallel SA$; (2) in the plane SAC, from the point M drop the perpendicular MN to AC; (3) from the point C drop the perpendicular CL to BD; (4) through the point N draw $NK \parallel CL$; (5) join the points M and K. The length of the line segment MK can be found from the right triangle MNK. **736.** $\frac{al^2\sqrt{3}\cos\alpha\sqrt{1+3\cos\alpha}}{2}$. **737.** $-\frac{a^2\cos\alpha\tan\beta}{2\cos 3\alpha}$. **738.** By $\frac{2S\sqrt{3}}{3}$ cm². **739.** $\frac{S(c+H)}{2}$.

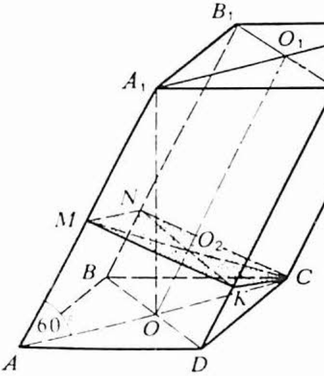


Fig. 211

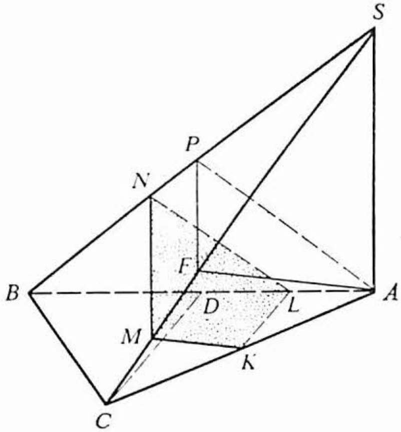


Fig. 212

740. $\frac{b}{6}\sqrt{a^2+c^2}$. **741.** $\frac{2aH}{\sqrt{9a^2+4H^2}}$. **742.** $\frac{a^2\sqrt{3}}{4\cos\alpha}$, where $a \leqslant \frac{2b\sqrt{3}}{3}\cot\alpha$ (b a lateral edge). **743.** $\frac{a^2\sin^2 2\alpha\cos\beta}{\sin^2(\alpha+\beta)}$. **744.** $\frac{a^2\sqrt{3}}{3}$. **745.** $\frac{a^2\sqrt{3}}{4}$. The required section can be constructed as follows (Fig. 211): (1) construct the median CM of the triangle AA_1C (obviously, then $CM \perp AA_1$); (2) find the point O_2 of intersection of the straight lines CM and OO_1; (3) in the plane BB_1D_1D, through the point O_2 draw the straight line $NK \parallel BD$; (4) the quadrilateral $CKMN$ is the desired section. **746.** $\frac{a^2}{2}\sqrt{\sin(60°+\alpha)\sin(60°-\alpha)}$, where $0° < \alpha < 45°$. **747.** $\frac{a^2\sqrt{3}}{4}\left(1+\sqrt{\frac{6n+3m}{m}}\right)$, or $\frac{a^2\sqrt{3}}{4}\left(1+\sqrt{\frac{6m+3n}{n}}\right)$, where $a < H\sqrt{6}$ (H the altitude of the pyramid). **748.** $\frac{a}{4}$. Since $AC=BC=a$ and $AB=a\sqrt{2}$ (Fig. 212), $\angle ACB=90°$, and in the right triangle SAC, $SC=a\sqrt{3}$, and in the right triangle SAB, $SB=2a$. The required section can be constructed in the following way: (1) construct CD, a median of the triangle ABC; (2) draw the straight line $KL \parallel CD$ (obviously, then $AB \perp KL$ and $SB \perp KL$); (3) construct AP, a median of the triangle SAB; (4) in the plane SAB, through the point L draw the straight line $LN \parallel AP$; (5) in the plane SAC, from the point A drop the perpendicular AF to the edge SC (for the construction it is possible to find that $CF:CS=1:3$); (6) in the plane SBC, through the point N draw the straight line $NM \parallel PF$; (7) join the points M and K. The quadrilateral $KLNM$ is the desired section. **749.** $\arctan\frac{\sqrt{5}}{4}$.

750. $\arctan\frac{4\sqrt{10}}{9}$. **751.** $\arctan\frac{\sqrt{21}}{3}$. **752.** $\arctan\frac{\sqrt{15}}{4}$.

753. $\arctan\left(\frac{\sqrt{3}}{3}\tan\alpha\right)$. **754.** $\arccos(\tan\alpha)$. **755.** $\frac{\tan\beta}{2\sin\alpha}$. **756.** 2:3.

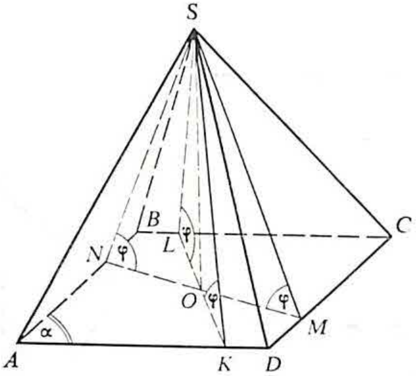


Fig. 213

757. $\arctan \frac{\sqrt{7}}{2}$. 758. $\arctan \frac{5\sqrt{13}}{6}$. 759. $\sqrt{6}$. 760. 11:12.

761. $\frac{a\sqrt{2a^2+4b^2}}{2a+\sqrt{2a^2+4b^2}}$. 762. $\frac{a^2 \sin^2 2\alpha \cos \alpha}{\sin^2 3\alpha}$. 763. $4d^2 \sin \alpha \sqrt{\cos 2\alpha}$.

764. $d^2 \sqrt{2} \sin 2\beta \cos (45° - \alpha)$. 765. $\frac{4\sqrt{2} H^2 \sin \frac{\alpha}{2}}{\sqrt{\cos \alpha}}$.

766. $\frac{6\sqrt{6} H^2 \cot \alpha \sin (45° + \alpha)}{\sin \alpha}$. 767. $\frac{\sqrt{2S \cos \alpha}}{2 \cos \frac{\alpha}{2}}$. 768. $\frac{4}{3} H^2$.

769. 96 cm², 144 cm², 48 cm², and 288 cm². 770. $\frac{\sin \alpha \sqrt{S \cos \alpha}}{2}$.

771. $\frac{1}{3} \sqrt{\frac{2S\sqrt{3} \sin \frac{\alpha}{2}}{\sin \left(\frac{\alpha}{2} + \frac{\pi}{3} \right)}}$. 772. $\frac{a^2 (1 + \sin \alpha + \cos \alpha)}{\cos \alpha}$.

773. $\frac{2H^2 \cos \frac{\alpha+\beta}{2} \cos \left(\frac{\pi}{4} + \frac{\beta}{2} \right) \cos \left(\frac{\pi}{4} - \frac{\beta}{2} \right)}{\sin \alpha \sin \beta}$. 774. $1 + \sqrt{2}$.

775. $\frac{4(1+\sin \alpha)\sqrt{\sin (\alpha+\beta) \sin (\alpha-\beta)}}{\sin 2\alpha \sin \beta}$. 776. $\frac{1+\sqrt{5}}{6}$. 777. $\frac{1+2\sin \alpha}{12 \sin \alpha}$.

778. $\frac{\pi}{\cos \alpha}$. 780. $\frac{\pi\sqrt{15}}{3}$. 781. $\arccos \frac{4}{5}$. 782. $2\pi a^2 \sqrt{3}$. 783. 270π cm².

784. $2\pi (a+b) \sqrt{a^2+b^2}$. 785. $\frac{2\pi a^2 \cos^2 \frac{\alpha}{2}}{\cos \alpha}$. 786. $\frac{4\pi R^2 (2 + \cos \alpha + 2 \sin \alpha)}{\sin \alpha}$.

787. 4π. 788. $\frac{\pi (5+4\sqrt{3})}{2}$. 789. $4\pi : \sqrt{3}$. 790. $\sqrt{2}+1$.

791. $a^2 \left(1+\sqrt{4+2\sqrt{2}}\right)$. 792. 120°. 793. $\left(\sqrt{S_1} + \frac{\sqrt{S_2}}{2} \right)^2$. 794. $\frac{a^2 \sin \alpha}{\cos \varphi}$. Let SO be

the altitude of the pyramid (Fig. 213). From the vertex S drop the perpendicular SN to the edge AB and the perpendicular SL to the edge BC. Then draw the straight lines NO and LO and denote their points of intersection with the sides CD and AD by M and K, respectively. Then $\angle SNO = \angle SLO = \angle SMO = \angle SKO = \varphi$ and $\triangle SON = \triangle SOL = \triangle SOM = \triangle SOK$, that is, $ON = OL = OM = OK$, and, consequently, the point O is equidistant from the sides of the base. 795. $\dfrac{d^2 \sin 2\alpha \cos^2 \frac{\varphi}{2}}{\cos\varphi}$. 796. $\dfrac{a^2\sqrt{3}}{3\cos\alpha}\left(1+\sqrt{1+3\sin^2\alpha}\right)$.

797. $2a^2 + 2a\sqrt{4b^2-a^2}$. 798. $ab(\sqrt{2}+1)$. 799. By three times. 800. $\dfrac{a^3}{2}$.

801. $2a^3 \sin\alpha \sqrt{\sin 3\alpha \sin\alpha}$. 802. $abc\sqrt{-\cos 2\alpha}$. 803. am^2.

804. $\dfrac{a^3 \sin\frac{\alpha}{2}\sin\alpha}{\sin\varphi}\sqrt{\cos\left(\frac{\alpha}{2}+\varphi\right)\cos\left(\frac{\alpha}{2}-\varphi\right)}$. 805. $\dfrac{a^2(b+c)}{2}$.

806. $\sqrt{3}\sqrt[3]{\dfrac{V^2}{\sin^2\alpha\cos\alpha}}$. 807. $\dfrac{a^3\sqrt{3}}{2}$. 808. $\dfrac{a^3\sin 2\alpha\sin\alpha}{2\cos^2\left(45^\circ-\frac{\alpha}{2}\right)}$.

809. $\dfrac{a^3}{3}\sin\dfrac{\alpha}{3}\sqrt{\sin\left(60^\circ+\frac{\alpha}{2}\right)\sin\left(60^\circ-\frac{\alpha}{2}\right)}$. 810. $\dfrac{2S}{3}\sqrt{\dfrac{S}{\tan\alpha}}$.

811. $\dfrac{H^3\sqrt{3}\sin\left(\frac{\varphi}{2}+30^\circ\right)\sin\left(\frac{\varphi}{2}-30^\circ\right)}{\cos^2\frac{\varphi}{2}}$. 812. $-\dfrac{2}{3}b^3\dfrac{\cos\frac{\varphi}{2}\cos\varphi}{\sin^3\frac{\varphi}{2}}$.

813. $\sqrt[3]{\dfrac{9V^2}{\tan^2\alpha}\dfrac{\sqrt{3}(1+\cos\alpha)}{\cos\alpha}}$. 814. $\dfrac{c^3}{24}\sin 2\alpha\tan\beta$.

815. $\dfrac{l^3(4+\tan^2\alpha)\sqrt{12+3\tan^2\alpha}}{8\tan^2\alpha}$. 816. $\dfrac{l^3\sqrt{3}(1+4\tan^2\beta)\sqrt{1+4\tan^2\beta}}{4\tan^2\beta}$.

817. $\dfrac{1}{12}b^3\sqrt{2}$. 818. $\dfrac{2PQ}{3b}$. 819. $\dfrac{H^3\sin^2(\alpha-\beta)}{3\sin^2\alpha\sin^2\beta}$. 820. $\dfrac{3\sqrt{2}}{4}a^2$.

821. 1:31. 822. $\sqrt[3]{\dfrac{9V^2}{\tan\alpha\tan\beta}}$. 823. $\dfrac{2}{3}a^2\cos^3\dfrac{\alpha}{2}\tan\beta$.

824. $\dfrac{1}{6}S\tan\beta\sqrt{S\sin\alpha}$. 825. $\dfrac{\pi bc(b+c)\sin\alpha\cos\frac{\alpha}{2}}{3}$. 826. $\dfrac{1}{6}\pi b^3(5+3\sqrt{2})$.

827. $2\pi S\sqrt[4]{\dfrac{4S^2}{27}}$. 828. 1:2. 830. $\dfrac{V_a}{V_b}=\dfrac{b}{a}$ and $\dfrac{V_b}{V_c}=\dfrac{c}{b}$. 832. $\dfrac{a^3-b^3}{6\cos\alpha}\times$ $\sqrt{-\cos 2\alpha}$. 834. $\dfrac{\pi R^3\sqrt{2}}{6}$. 835. $\dfrac{R^3\alpha^2\sqrt{4\pi^2-\alpha^2}}{24\pi^2}$. 836. $\dfrac{a^3}{48}(4\sqrt{2}+\sqrt{10})$.

837. $\dfrac{\pi aS}{\sin\alpha}$. 838. 1:4. 839. $\dfrac{3\sqrt{3}+1}{26}$. 840. 97:191. 841. 25:47. 842. 1:47. 843. 7:29. 844. 7:17. 845. 13:23. 846. 7:29. 847. 8:37. 848. 1:14. 849. 3:5. 850. $\dfrac{\sqrt{2}}{3}$ and $\dfrac{5\sqrt{2}}{7}$. 851. 3:1. 852. 5:24.

853. $2a\sqrt{R^2-d^2}$. 854. $\sqrt{\dfrac{a^2}{8}+\dfrac{R^2}{2}}$, where $a \leqslant 2R$. 855. $-\dfrac{1}{3}\pi R^3\tan^3\alpha\tan 2\alpha$,

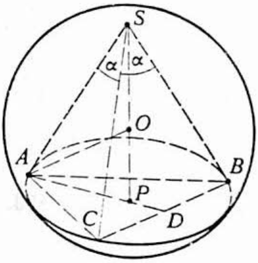


Fig. 214

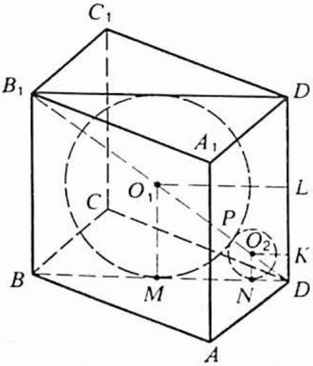


Fig. 215

where $90° < \alpha < 112°30'$. **856.** $\pi R^2(1+\cos^2 2\alpha + 2\sin\alpha\sin 2\alpha)$. **857.** $\sqrt{\frac{S_1(S_2-S_1)}{\pi(S_2+S_1)}}$.

858. $\frac{\pi R^3 \cot\left(45° - \frac{\alpha}{2}\right)}{\cos^3\alpha}$. **859.** $\arccos\left(-\frac{1}{3}\right)$. **860.** $2\arccos\left(\frac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{4}}\right)$.

861. $\arccos(\sqrt[3]{2}-1)$. **862.** 1:1 and 9:7. **863.** $2\arcsin\frac{m-n}{m+n}$. **865.** 60°. **866.** $\frac{2\cos^2\alpha}{\cos 2\alpha\tan\alpha}$. **867.** $\arcsin\frac{n-m}{n+m}$. **868.** $\sqrt{3}:1$. **869.** $2\arctan\frac{1}{\sqrt{n}}$.

870. cot 40°:cot 30°:cot 20°. **872.** $\arctan\sqrt{2}$. **873.** $2R\sqrt{1-\frac{4}{3}\sin^2\frac{\alpha}{2}}$ (Fig. 214). Since $SA=SB=SC$ and $\angle ASB = \angle BSC = \angle CSA = \alpha$, $SABC$ is a regular triangular pyramid. Let $SC = x$. Then $CD = x\sin\frac{\alpha}{2}$, $BC = 2x\sin\frac{\alpha}{2}$, $AD = \sqrt{3}\,x\sin\frac{\alpha}{2}$, and $AP = \frac{2}{3}AD = \frac{2}{3}\sqrt{3}\,x\sin\frac{\alpha}{2}$.

874. $\frac{a\sin\alpha}{2\left(\sin\frac{\alpha}{2}\cot\frac{\beta}{2}+1\right)}$. **875.** $\frac{24+7\pi}{48}$. **876.** $\frac{1}{12}\pi a^3(15-8\sqrt{2})$.

877. $\frac{1}{4}a(3-\sqrt{3})$. **878.** $\frac{a(2-\sqrt{3})}{2}$. Let both balls touch the faces having a common vertex D (Fig. 215). Then the points O_1 and O_2, the centres of these balls, lie on the diagonal B_1D of the cube. Since the larger ball touches both the upper and lower bases, $O_1M = \frac{a}{2}$ and $O_1L = MD = \frac{a\sqrt{2}}{2}$. Setting, for brevity, $O_2N = x$, get from the similarity of the triangles O_2KD and O_1LD that $O_2K = x\sqrt{2}$ and, further, $O_2D = x\sqrt{3}$. **879.** $\frac{1}{8}a\sqrt{41}$.

880. $\frac{1}{2}R(2+\sqrt{2})$. **881.** $\frac{1}{2}a(3-\sqrt{3})$. **882.** $a(3\sqrt{2}-2\sqrt{3})$. **883.** $\frac{20\sqrt{3}}{27}R$.

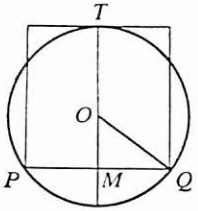


Fig. 216

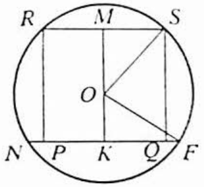


Fig. 217

884. 17:125. Let the ball touch the upper base of the cube at the point T and the edges AB, BC, CD, and DA of the lower base. The points of contact of the ball and the edges are the midpoints of these edges. Consider the section of the ball and cube by the plane TPQ, where P is the midpoint of the edge AB, and Q is the midpoint of the edge CD (Fig. 216). This section passes through the point O, i.e. the centre of the ball. Introduce an auxiliary parameter $OT=R$ and set, for brevity, the length of the edge of the cube to be equal to x. Then from the triangle OMQ, $x=\frac{8}{5}R$. There are five segments of the ball outside the cube: one under the base of the cube and four (equal to one another) at the lateral faces of the cube. 885. $8:\sqrt{41}$. Consider the section of the ball and cube by the plane ORS passing through the points R and S, i.e. the midpoints of two opposite edges of the upper base of the cube, and the point O, i.e. the centre of the ball (Fig. 217). (The points P and Q, i.e. the midpoints of two opposite edges of the lower base, "hang", since the radius of the ball is longer than half the length of the edge of the cube.) Introduce an auxiliary parameter, setting the length of the edge of the cube equal to a. Then $MF=a\sqrt{2}$. Further, consider the triangles OKF and OMS. 886. $\frac{1}{7}\pi(5\sqrt{2}-6)$. 887. $\dfrac{h^3}{\sqrt{2}\sin 2\alpha\cos\frac{\alpha}{2}\cos\left(45^\circ-\frac{\alpha}{2}\right)}$.

888. $2\arctan\frac{1}{2}$. 889. $\arcsin\frac{6}{\pi m}$ and $\pi-\arcsin\frac{6}{\pi m}$. 890. $\frac{a\sqrt{2}}{2}$. 891. $\frac{a\sqrt{6}}{8}$.

893. $\dfrac{a\tan\alpha}{2\sqrt{3-4\sin^2\alpha}}$. 894. $\dfrac{2\sqrt{3}\,H\tan\frac{\alpha-\pi}{6}}{9\tan^2\frac{\alpha-\pi}{6}-3}$. 895. $\dfrac{\sqrt{16S^2+45a^4}}{24a}$.

896. 1:1 and 9:7. 897. $\frac{4}{3}R^3\dfrac{\cot^3\frac{\alpha}{2}\tan\varphi}{\sin\alpha}$. 898. $\dfrac{a^4}{48\sqrt{b^2-\frac{a^2}{2}}}$.

899. $\frac{1}{2}\tan\left(45^\circ-\frac{\alpha}{4}\right)\sqrt[3]{6V\tan\frac{\alpha}{2}}$. 900. $\frac{1}{2}\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$. 901. $2\sqrt{2}R^2\times\cos\alpha\left(\sin\alpha+\sqrt{1-\frac{1}{2}\cos^2\alpha}\right)$. 902. $\dfrac{\pi H\alpha\sqrt{\cos\alpha}}{45^\circ\cos\frac{\alpha}{2}}$. 903. At the edge of the base, $\arccos\frac{n-1}{n+2}$, and at the lateral edge, $2\arctan\sqrt{\frac{n^2+1}{2n}}$.

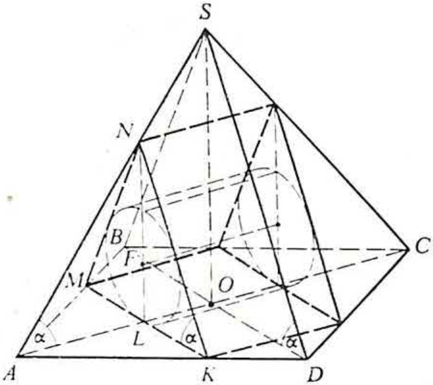


Fig. 218

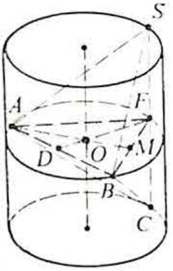


Fig. 219

904. $\dfrac{\sin\left(\frac{\pi}{3}-\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{3}+\frac{\alpha}{2}\right)}$. **905.** $\sqrt{2}\,b\sin\frac{\alpha}{2}\sin\left(45^\circ-\frac{\alpha}{2}\right)$ (Fig. 218). Let $FL=x$, then $OL=x$ too. Find from the triangle MNK (in which the base circle of the cylinder is inscribed) that $LK=x\cot\frac{\alpha}{2}$, and from the triangle SAO that $AO=b\cos\alpha$. Then $AL=b\cos\alpha-x$, but $AL=LK$, that is, $b\cos\alpha-x=x\cot\frac{\alpha}{2}$. **906.** $\dfrac{H^2\sin\frac{2\alpha+\beta}{2}\cos\frac{2\alpha-\beta}{2}}{\sin^2\alpha}$. **907.** $\frac{3\sqrt{2}}{8}a$. Consider the section of the tetrahedron by the plane passing through the vertices A, B and the point F, i.e. the midpoint of the edge SC (Fig. 219). In the triangle ABF, $AB=a$. Then $AF=BF=\frac{a\sqrt{3}}{2}$ and $DF=\frac{a\sqrt{2}}{2}$. Since the triangle ABF is inscribed in a circle, the point O is the point of intersection of the perpendiculars drawn to the midpoints of its sides. Find from the similarity of the triangles FMO and FBD (M is the midpoint of the side BF) that $\frac{OF}{BF}=\frac{MF}{DF}$. **908.** 6 cm. **909.** $\dfrac{9R^3\tan\alpha}{4(\sqrt{3}+\tan\alpha)^3}$. **910.** $l=\dfrac{a\sqrt{3}}{\cos\frac{\alpha}{2}+\sqrt{2}\sin\frac{\alpha}{2}}$ and $R=\dfrac{a\sqrt{3}}{\cot\frac{\alpha}{2}+\sqrt{2}}$. Consider the diagonal section of the cube. It is also an axial section of the cone inscribed in the cube (Fig. 220) (the point M does not lie on the edge DD_1, since the cone does not touch the edges of the cube). It is clear that $BB_1=a$, $BD=a\sqrt{2}$, and $B_1D=a\sqrt{3}$. Let $PQ=x$. Then from the right triangle B_1PQ, $B_1Q=x\cot\frac{\alpha}{2}$, and from the similarity of the triangles DPQ and DBB_1, $\frac{PQ}{BB_1}=\frac{DQ}{BD}$. **911.** $\frac{R^2\sqrt{6}}{4}$. **912.** $\dfrac{\pi H^3\sin(2\alpha-\beta)\sin\beta}{3\sin^2\alpha\sin^2(\alpha-\beta)}$.

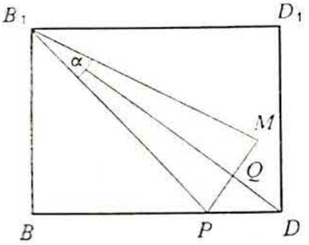


Fig. 220

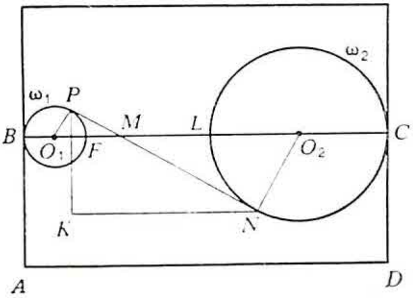
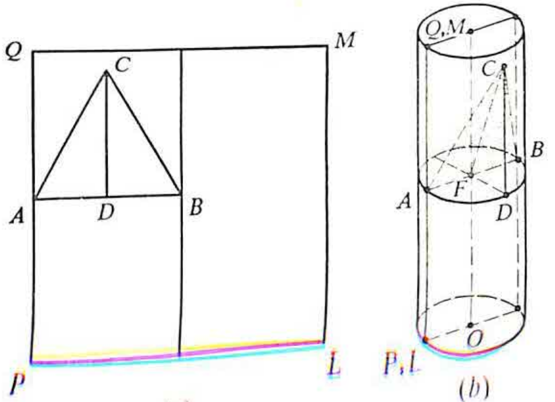


Fig. 221

913. $\arccos \frac{\sqrt{ab}}{h}$. 914. $\frac{a\sqrt{12+\pi^2}}{2\sqrt{3}\pi}$ (Fig. 221). Since $BC=BO_1+O_1O_2+O_2C=\frac{a}{12}+\frac{2a}{3}+\frac{a}{4}=a$ and $AD=a$, $BC=AD$, i.e. the circles ω_1 and ω_2 touch the lateral sides of the rectangle. It follows from the similarity of the triangles O_1MP and O_2MN that $\frac{O_1M}{O_2M}=\frac{O_1P}{O_2N}$, whence $O_2M=\frac{a}{2}$. But $O_2N=\frac{a}{4}$, then $\angle O_2MN=30°$, and, consequently, after constructing the straight line $KN \parallel BC$ and the straight line $PK \parallel AB$, find in the triangle PNK that $\angle PNK=30°$. Further, $O_1M=\frac{a}{6}$, then $PM=\frac{a\sqrt{3}}{12}$ and $MN=\frac{a\sqrt{3}}{4}$, i.e. $PN=\frac{a\sqrt{3}}{3}$, and from the triangle PKN, $KN=\frac{a}{2}$, whence $PK=\frac{a\sqrt{3}}{6}$. 915. $\frac{b^3\sqrt{5}}{150}$. 916. $\frac{5a^3}{48}$. 917. $\left(\frac{abc}{ab+ac+bc}\right)^3$. 918. $\frac{b\sqrt{2}\sin 2\alpha}{4\sin(45°+\alpha)}$. 919. $\frac{8\pi^2}{\sqrt{3\pi^2+4}}$ (Fig. 222a, b). Introduce an auxil-



(a) (b)

Fig. 222

iary parameter, setting $PL=2a$. Then $AB=a$. Since the square $PQML$ is rolled up to form a cylinder, $2\pi OP=2a$, whence $OP=\frac{a}{\pi}$, and, consequently, $AB=\frac{2a}{\pi}$. When the square is rolled up, the length of the line segment CD remains unchanged, i.e. $CD=\frac{a\sqrt{3}}{2}$. Further, from the triangle CFD, $CF=\frac{a}{2\pi}\sqrt{3\pi^2+4}$. 921. (a) $\sqrt{2}$ cm³; (b) 6 cm². 922. $\frac{P\sqrt{5}}{10}$. 923. $3\sqrt{3}$ cm. 924. $\arctan\frac{\sqrt{2}}{2}$. 925. $\arctan\sqrt{2}$. 926. 8 m. 928. 2 cm. 929. $3\sqrt{3}$ cm² and 12 cm². 932. RH if $R\leqslant H$ and $\frac{R^2+H^2}{2}$ if $R>H$. 933. $\sqrt{\frac{3V}{5\pi}}$. 934. (a) $\frac{p}{2}$, $\frac{3p}{4}$, and $\frac{3p}{4}$; (b) $\frac{3p}{5}$, $\frac{3p}{5}$, and $\frac{4p}{5}$. 935. $2\pi\sqrt{\frac{2}{3}}$. 936. $h=\frac{H}{3}$, where h is the altitude of the cylinder, and H is the altitude of the cone. 937. (a) $\frac{2R\sqrt{3}}{3}$; (b) $R\sqrt{2}$. 938. (a), (b) $\frac{4R}{3}$. 939. (a) $4R$; (b) $R\sqrt{2}$. 940. $R\sqrt{3}$. 941. $\frac{8}{27}(\sqrt{3}-1)^3$. 942. $\frac{4}{9}$. 943. 45°. 944. $\frac{1}{3}$. 945. $\frac{R\sqrt{13}}{13}$. 946. $\frac{32\sqrt{3}}{27}R^3$. 947. $\arctan\left(\cos\frac{\pi}{n}\sqrt{2(\sqrt{2}+1)}\right)$. 948. $R=\frac{9a}{4}$ and $H=3a$. 949. $\frac{16\sqrt{3}}{3}$ cm³. 950. $\frac{16}{81}V$. 951. $\frac{1}{12}V$.

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Selected Problems and Theorems in Elementary Mathematics

Arithmetic and Algebra

By **D.O. Shklyarsky**,
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I.M. Yaglom, D.Sc. (Phys.-Math.)

This book contains 350 problems in algebra, arithmetic and number theory selected from the material of the mathematical olympiads and school mathematics hobby groups in Moscow. About 15 problems have been taken from the manuscript of the late D.O. Shklyarsky, one of the founders of the mathematics hobby group for pupils at Moscow State University. The methods of solving the problems are described in detail, while the answers and hints to the most difficult problems are given at the end of the book.

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